PUMaC 2011

Combinatorics A Solutions

1. Since there is no carrying involved, we can do casework based on the sum's units digit. There are no sums which have a units digit of $0, 1$, or 2. If it is 3, 4, 8, or 9 then we know which two digits were added; in each of these cases there are three possible values for the sum's tens digit, after which the hundreds digit is determined. If it is 5, 6, or 7 then there are two possible pairs of added digits, and it is easily seen that in every such case there are five possible values for the sum's tens digit. Therefore there are $4 \cdot 3 + 3 \cdot 5 = \boxed{27}$ possible values for the sum.

2. Let c_n denote the number of such colorings. If the rightmost column of two squares have the same color (2 ways), then those two squares cannot be occupied by the same domino, so each must be covered by a horizontal domino. Then, the previous column must be two squares of the opposite color, and the rest of the $2 \times (n-2)$ board can be colored in c_{n-2} ways. If they have two different colors (2 ways), then one can suppose that the rightmost column is covered by a vertical domino, so that the rest of the $2 \times (n-1)$ board can be colored in c_{n-1} ways: this is only because if the rightmost column is covered by two horizontal dominos, then we can rotate those two dominos and result in a configuration where the rightmost column is covered by a vertical domino, and we still have the same coloring. Hence we have the recursion $c_n = 2c_{n-1} + 2c_{n-2}$ with $c_1 = 2, c_2 = 6$, and we compute that $c_6 = \boxed{328}$.

   Challenge: What if instead we have an infinite supply of two types of dominoes, one of which has one white square and one black square, and the other which has two black squares?

3. Fix one edge for the first point to lie on. If the second point lies on the opposite edge, it will be of distance greater than one (with 1/4 probability), and if it lies on the same edge, then it will be of distance less than one (again 1/4 probability). Suppose then that the second point lies on one of the other two adjacent edges. If the first point is distance x from this edge, then the other point must lie farther than $\sqrt{1-x^2}$ from the vertex shared by the edges the two points lie on. Therefore, this reduces to a geometric probability problem where we are finding the area of a region outside of a quarter circle in the unit square (we could also view this as the integral $\int_0^1 \sqrt{1-x^2}\,dx$, although this is a bit more work). Thus, $p = \frac{1}{4} + \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{6-\pi}{8}$, and $\lfloor 100p \rfloor = \boxed{35}$.

4. To find the answer, we can instead subtract the number of ways we can position 4 bishops such that at least 3 are bishops on a diagonal from the total number of cases. Also, since the problem asks for the remainder of the answer divided by 100, we only need to keep track of the last two digits for intermediate steps. There are in total $\binom{25}{4} \equiv 50 \pmod{100}$ ways of placing 4 bishops on the board.
   • When there are 4 bishops on a diagonal, they are either on the main diagonals or the diagonals with length 4. There are
   $$2\binom{5}{4} + 4 = 14$$
   cases of this kind.
   • Otherwise, there are exactly 3 bishops on some diagonal. They can be on the main diagonals,

the diagonals with length 4, or the diagonals with length 3. The number of cases can also be calculated:

$$2\binom{5}{3}\binom{20}{1}+4\binom{4}{3}\binom{21}{1}+4\binom{22}{1}\equiv 0+16\cdot 21+88\equiv 24 \pmod{100}.$$

Therefore, we obtain the final answer to be $50-14-24=\boxed{12}$.

5. Suppose in general that σ is a permutation of a set of size $n>1$. Let $P[L(\sigma)=l]$ be the probability that $L(\sigma)$ is equal to l, and define $\mathbb{E}[L(\sigma)]$ to be the expected value of $L(\sigma)$. By definition of expected value,

$$\mathbb{E}[L(\sigma)]=\sum_{m=0}^{n} m\cdot P[L(\sigma)=m]=\sum_{m=1}^{n}\sum_{l=1}^{m} P[L(\sigma)=m]=\sum_{l=1}^{n}\sum_{m=l}^{n} P[L(\sigma)=m].$$

Here, $\sum_{m=l}^{n} P[L(\sigma)=m]$ is exactly the probability $P[L(\sigma)\geq l]$ that $L(\sigma)$ is at least l. We can calculate that $P[L(\sigma)\geq l]=\frac{2(n-l)!(n-l)}{n!}$: after choosing between increasing and decreasing, there are $n-l$ possible starting values for our initial sequence of length l, and then we can choose any of the $(n-l)!$ arrangements of the last $n-l$ elements of the permutation. Then, we have that

$$\mathbb{E}[L(\sigma)]=\sum_{l=1}^{7} P[L(\sigma)\geq l]=\frac{n-1}{n}+2\sum_{l=2}^{n}\frac{(n-l)!(n-l)}{n!}.$$

Here,

$$\sum_{l=2}^{n}(n-l)!(n-l)=\sum_{l=1}^{n-2} l\cdot l!=\sum_{l=1}^{n-2}(l+1)!-l!=(n-1)!-1$$

by telescoping. It follows that

$$\mathbb{E}[L(\sigma)]=\frac{n-1}{n}+\frac{2\cdot(n-1)!-2}{n!}=\frac{(n+1)(n-1)!-2}{n!}.$$

For $n=7$, this yields $\frac{5758}{5040}=\frac{2879}{2520}$ (note $\gcd(2879,2520)=\gcd(359,2520)=\gcd(359,7)=1$ by the Euclidean algorithm, so this is reduced), so the answer is $\boxed{5399}$.

6. **First solution**: Turning to generating functions, this is the same problem as asking for the coefficient of x^{263} in the polynomial

$$(1+x+x^2+x^3)(1+x^2+x^4+x^6)\cdots(1+x^{64}+x^{128}+x^{192})$$

and we can simplify the polynomial by telescoping:

$$\begin{aligned}\frac{x^4-1}{x-1}\cdot\frac{x^8-1}{x^2-1}\cdots\frac{x^{256}-1}{x^{64}-1}&=\frac{x^{128}-1}{x^2-1}\cdot\frac{x^{256}-1}{x-1}\\&=(1+x^2+x^4+\cdots+x^{126})(1+x+x^2+\cdots+x^{255}).\end{aligned}$$

From here we can easily calculate the coefficient of x^{263} to be $\boxed{60}$.

Second solution: First, notice the sum of all of the weights is $127 \times 3 = 381$, and any combination of weights that sum to 263 corresponds to a combination of weights that sum to $381 - 263 = 118$ (by using the leftover weights instead). So we instead consider the number of ways of forming a total weight of 118. Then, we can construct a bijection between any set of weights to the set of even nonnegative integers $2m \leq 118$: use twice the weights corresponding to the binary expansion of m, and then add the weights corresponding to the binary expansion of $118 - 2m$. This is possible because the binary expansion of any natural number less than or equal to 118 has at most 7 binary digits, corresponding to $2^6 = 64$. Conversely, for any such combination of weights, if zero or one of the 2^n weights are used, then the corresponding binary digit for m will be 0, while if two or three of the 2^n weights are used, then the corresponding binary digit for m will be 1. So again, there are $\frac{118}{2} + 1 = \boxed{60}$ ways in total.

7. Define a_i such that if in the ith minute from the beginning someone enters, then $a_i = 1$, and if someone leaves, then $a_i = -1$. Hence, each possible sequence of entries and exits is denoted by a sequence $\{a_i\}_{i=1}^{200}$ containing 100 values of 1 and 100 values of -1. Define $S_n = \sum_{i=1}^{n} a_i$ for $0 \leq n \leq 200$, and $M = \max_{1 \leq i \leq 200} S_i$. First consider the number of possible sequences for which $M \geq 10$.

   If a sequence $\{a_n\}$ has $M \geq 10$, we can find i such that $S_i = 10$, since $S_0 = 0$ and every S_i differs by 1 from the previous one. Let p be the minimum of these such i's, so that $S_p = 10$ and $S_i < 10$ for any $i < p$. Now define another sequence $\{b_i\}$ such that

$$b_i = \begin{cases} -a_i & \text{if } i \leq p, \\ a_i & \text{if } i > p \end{cases}.$$

   Then, this forms a bijection between all such sequences $\{a_n\}$ with $M \geq 10$ to all such sequences $\{b_i\}_{i=1}^{200}$ containing 90 values of 1 and 110 values of -1's, since it is not hard to check that this "reflection" is one-to-one and onto. (Essentially what we are doing is noting that for any random walk with 200 steps of ± 1 starting at 90, ending at 90, and going above 100, then we can reflect the sequence after the first time the sequence goes above 100, which gives a random walk starting at 90 and ending at 110.) Here, the number of sequences $\{b_i\}$ is $\binom{200}{90}$, so the number of $\{a_n\}$ with $M \geq 10$ is also $\binom{200}{90}$.

   Similarly, the number of $\{a_i\}$ with $M \geq 11$ is $\binom{200}{89}$. Subtracting the two gives the number of sequences $\{a_i\}$ with $M = 10$. This exactly satisfies the requirement of the problem, since the maximum number of people at any time is $90 + 10 = 100$. Also, the number of people will not be negative, otherwise the number of people cannot reach 100 at any time. So

$$n = \binom{200}{90} - \binom{200}{89} = \frac{21 \times 200!}{90! \times 111!}.$$

   The greatest power of 2 in a factorial $k!$ is given by Legendre's formula to be

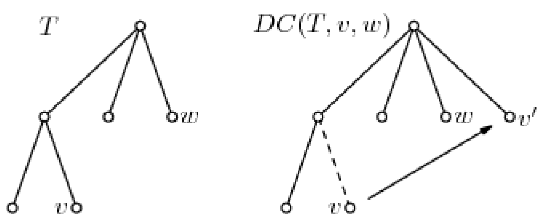
$$v_2(k!) = \sum_{i=1}^{\lfloor \log_2 k \rfloor} \left\lfloor \frac{k}{2^i} \right\rfloor,$$

so calculating the powers in these three factorials, $m = v_2(200!) - v_2(90!) - v_2(111!) = 197 - 86 - 105 = \boxed{6}$.

8. This solution is based on a proof in (Walks and Paths in Trees, Bollábos and Tyomkyn, 2011), which also discusses a number of other extremal results about walks and paths in trees.

For any tree T (a tree is an acyclic undirected graph), define $P_k(T)$ to be the number of k-paths (a k-path is a sequence of $k+1$ distinct vertices, for which there is an edge between consecutive vertices) in T. Consider any tree T with $P_4(T)$ maximal, given that it has $|E(T)| = 20$ edges; then the problem asks for the value of $2P_4(T)$.

For any leaves $v, w \in V(T)$, define $n_k(v)$ to be the number of vertices of T which are of distance k from v, which is the same as the number of k-paths containing v, and define $DC(T, v, w)$ to be the tree resulting from deleting v, and attaching a leaf to the unique neighbor of w (the delete-clone operation). Thus $|E(DC(T, v, w))| = |E(T)|$.



First suppose v, w are leaves such that $d(v, w) \neq 4$, so that there are no 4-paths containing both v and w, and without loss of generality suppose $n_4(v) < n_4(w)$. Then, $P_4(DC(T, v, w)) = P_4(T) + n_4(w) - n_4(v) > P_4(T)$, contradicting the maximality of $P_4(T)$. Hence, for any leaves v, w not of distance 4 apart, then $n_4(v) = n_4(w)$, and $P_4(T) = P_4(DC(T, v, w))$.

Now, we show that we can move around vertices so that all leaves are of distance 2 or 4 apart, without decreasing the number of 4-paths. If v is a leaf, let the leaf class of v be the set of all leaves adjacent to the unique neighbor of v, e.g., all leaves of distance 2 from v. Then, if v, w are leaves with $d(v, w) \neq 2, 4$, recursively define $T_0 = T$, $T_{n+1} = DC(T_n, v, w')$ where w' is a leaf in the leaf class of w; this merges the leaf classes of v and w. Hence, the number of leaf classes is a decreasing invariant in that if T is a tree with $P_4(T)$ maximal and a minimal number of leaf classes, then any leaves are of distance 2 or 4 apart. (It is easy to see this argument generalizes to k-paths instead of 4-paths.)

Let a star S_p centered at v be the graph obtained by attaching p leaves to v. If the leaves of T are either distance 2 or 4 apart, then T can be constructed from a star S_m centered at v, each of whose leaves v_i is replaced by another star S_{p_i} centered at v_i (so each v_i has $p_i + 1$ neighbors). Then, we find that $P_4(T) = \sum_{i<j} p_i p_j$. For any s, t, then we can write

$$P_4(T) = p_s p_t + (p_s + p_t) \sum_{i \neq s,t} p_i + \sum_{\substack{i<j \\ i,j \neq s,t}} p_i p_j,$$

which (say by AM-GM) for any fixed value of $p_s + p_t$ is maximized when p_s, p_t are as close to each other as possible. Hence, all of the p_i's are within one of each other, so the tree T is uniquely determined by m. Trying these possible values ($m = 2, 3, \ldots$), we see that $P_4(T)$ is maximized when $m = 4$, and the answer is $2P_4(T) = 2 \cdot \binom{4}{2} \times 4^2 = \boxed{192}$.