Number Theory A Solutions

1. The sum of the divisors of $n = 2^i 3^j$ is equal to $(1+2^1+2^2+\cdots+2^i)(1+3^1+3^2+\cdots+3^j) = 1815$, since each divisor of 2^i3^j is represented exactly once in the sum that results when the product is expanded. Let $A = 1+2^1+2^2+\cdots+2^i = 2^{i+1}-1$ and $B = 1+3^1+3^2+\cdots+3^j$, so that $AB = 1815 = 3\cdot 5\cdot 11^2$.

   Since $B \equiv 1 \pmod 3$, $3|A$. By Fermat's Little Theorem, $2^{i+1}-1 \equiv 0 \pmod 3$ only when i is odd. For $i = 1$ we get $A = 3, B = 605$ which does not work. For $i = 3$ we get $A = 15, B = 121$, which holds for $j = 4$ and $n = \boxed{648}$. For $i = 5, 7,$ and 9, we obtain $7|A, 17|A,$ and $31|A$ respectively (all of which do not divide 1815), and for $i > 10$, $A > 1815$.

2. Using the identity that $\operatorname{lcm}(m,n)\cdot\gcd(m,n) = m\cdot n$, it follows that

$$3m \times \gcd(m,n) = \operatorname{lcm}(m,n) = \frac{m\cdot n}{\gcd(m,n)} \quad\Longrightarrow\quad n = 3[\gcd(m,n)]^2.$$

   It follows that n must be three times a perfect square. If we set $m = \sqrt{n/3}$, which is an integer, it follows that

$$\operatorname{lcm}(\sqrt{n/3}, n) = n = 3\sqrt{n/3}\cdot\sqrt{n/3} = 3\sqrt{n/3}\cdot\gcd(\sqrt{n/3}, n),$$

   as desired. Hence, every triple of a perfect square works as a value of n, and the largest such under 1000 is $3\cdot 18^2 = \boxed{972}$.

3. Note that $7^3 = 343 \equiv -1 \pmod{43}$ and that $6^6 = (6^3)^2 \equiv 1 \pmod{43}$. Therefore, for $p \equiv 0,1,2,3,4,5 \pmod 6$, $7^p - 6^p + 2 \equiv 2,3,15,0,32,3 \pmod{43}$. Therefore, if $43|7^p - 6^p + 2$, $p \equiv 3 \pmod 6$. This means that $p = \boxed{3}$ is the only solution, so that the sum of all solutions is 3.

4. Let the set be $\{a,b,c\}$, and without loss of generality, suppose that $a \le b \le c$. Then

$$abc - 2a - 2b - 2c = 4.$$

   If $c \ge b \ge a \ge 4$, then

$$4 = abc - 2a - 2b - 2c \ge 16c - 2c - 2c - 2c = 10c \ge 40,$$

   which is a contradiction. Thus, $a \in \{1,2,3\}$.

   If $a = 1$, we get that $bc - 2b - 2c = 6$. Completing the rectangle, $bc - 2b - 2c + 4 = (b-2)(c-2) = 10$. Thus, $b-2$ and $c-2$ are a pair of (positive) factors of 10, and so must be equal to $1, 10$ or $2, 5$. Thus, we get the solutions $\{a,b,c\} = \{1,3,12\}$ and $\{1,4,7\}$, which can be permuted in 12 ways (6 for each solution).

   If $a = 2$, we get that $2bc - 2b - 2c = 8$, so $bc - b - c + 1 = (b-1)(c-1) = 5$. Thus, $b-1$ and $c-1$ are a pair of factors of 5, and so must be equal to $1, 5$. This gives the solution $\{a,b,c\} = \{2,2,6\}$, which can be permuted in 3 ways.

If $a = 3$, we get that $3bc - 2b - 2c = 10$, so $9bc - 6b - 6c + 4 = (3b - 2)(3c - 2) = 34$. Then $3b - 2, 3c - 2$ are a pair of factors of 34 and $(3b - 2, 3c - 2) = (1, 34), (2, 17)$. In the first pair of solutions, we have $b = 1 \le a = 3$, and in the second pair of solutions, we do not get integer values for b, c.

In total, we have $\boxed{15}$ ordered triplets which satisfy the given conditions.

5. First, show that this sum converges by showing that $d(n) \le 2\sqrt{n}$ for all $n \ge 1$. Let x be some divisor of n with $x \le \sqrt{n}$. There is a one to one correspondence between divisors at most $\sqrt{n}$ and divisors at least $\sqrt{n}$ (map x to $\frac{n}{x}$ for any $x \le n$). However, there are at most $\sqrt{n}$ divisors of n that are at most $\sqrt{n}$, so $d(n) \le 2\sqrt{n}$ for all positive integers n. Therefore, the given sum is bounded above by

$$\sum_{n=1}^{\infty} \frac{2}{x^{\frac{3}{2}}}$$

which converges.

First solution: Note that $d(n) = \sum_{k|n} 1$, so $\dfrac{d(n)}{n^2} = \sum_{k|n} \dfrac{1}{k^2(n/k)^2}$. This means that the desired sum is equal to

$$\sum_{n=1}^{\infty} \sum_{k|n} \frac{1}{k^2(n/k)^2} = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{k^2 m^2}.$$

This can be seen by letting $m = \frac{n}{k}$ and switching the order of summation (which is valid by the absolute convergence of the original series). Therefore, we can write this series as

$$\left(\sum_{m=1}^{\infty} \frac{1}{m^2} \right)^2 = \frac{\pi^4}{36}.$$

Hence $p(x) = \frac{x^4}{36}$ and $p(6) = \boxed{36}$. $p(x)$ is unique because π is transcendental.

Second solution: Since the original sum converges, one may use the unique prime factorization of the integers and the fact that $d(n)$ is multiplicative to factor the given sum as

$$\prod_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{d(p_i^j)}{p_i^{2j}}$$

where p_i is the ith prime. Note also that $d(p^j) = j + 1$ for any prime p. Now, compute the sum

$$S(x) = \sum_{j=0}^{\infty} (j + 1) x^j$$

To compute a closed form for $S(x)$, consider $xS(x) = \sum_{j=1}^{\infty} j x^j$ and $S(x) - xS(x) = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ for $|x| < 1$. Therefore, $S(x) = \frac{1}{(1-x)^2} = \left(\sum_{j=0}^{\infty} x^j \right)^2$. Letting $x = \frac{1}{p_i}$ for each term of the infinite product shows that the desired sum is equal to $(\sum_{j=1}^{\infty} \frac{1}{j^2})^2 = \frac{\pi^4}{36}$. Hence $p(x) = \frac{x^4}{36}$ and $p(6) = \boxed{36}$.

6. In order to decrease the number of remainders $\pmod b$ that can be written in the form x^3+y^4, we should minimize the number of cubes and fourth powers $\pmod b$. This happens when b is prime and when $3, 4|b-1$, which happens when $b = 13$. Indeed, for this value of b, the cubic residues are $0, 1, 5, 8$ and 12 and the quartic residues are $0, 1, 3$, and 9. 7 cannot be written as the sum of any cubic-quartic residue pair, so $(7, 13)$ satisfies the problem's constraints.

   Now, I will show that for any $b < 13$, all residues $\pmod b$ can be written as a sum of a cubic-quartic residue pair. We can reduce the checking to just prime values of b and powers of primes by the Chinese Remainder Theorem. Therefore, we just need to check $b = 1, 2, 3, 4, 5, 7, 8, 9, 11$. 1, 2, and 3 are trivial to check, so just look at 4, 5, 7, 8, 9, 11. For 5, note that $x^4 \equiv 1$ for nonzero x, so $x^3 \equiv x^{-1}$. Everything is invertible $\pmod 5$ (except 0), so x^3 takes on all values $\pmod 5$. Very similar logic applies for 11, as $x^9 \equiv x^{-1} \pmod{11}$ for all nonzero x, which means that 9th powers take on all values $\pmod{11}$ (and therefore implies that the cubes take on all values $\pmod{11}$). For 7, note that $x^4 = x^{-2} \pmod 7$ for all nonzero x. This means x^4 takes on the values of all squares (which are 0,1,2,4). The cubes equivalent to $1, 0, -1 \pmod 7$, so just a little bit of checking shows that all residues modulo 7 are covered.

   Now, we just have to check $b = 4, 8, 9$. 4 is easy to check. By Euler's Theorem, $x^4 \equiv 1 \pmod 8$ for all odd residues $\pmod 8$, so $x^3 \equiv x^{-1}$ and the cubes cover all odd residues mod 8. Since the quartic residues are 0 and 1, we cover everything. Finally, we just need to check $b = 9$. For all invertible elements x, $x^4 \equiv x^{-2}$, so we cover all squares. The squares $\pmod 9$ are 0, 1, 4, 7. The cubes contain 0, 1, and -1 so we hit everything. Therefore, $(7, 13)$ is the solution with smallest b. Thus the answer is $7 \cdot 13 = \boxed{91}$.

7. This sequence, which starts off as $1, 1, 1, 2, 3, 5, 21, 34, \ldots$ contains many members of the Fibonacci sequence. However, it is not the Fibonacci sequence. If $\{F_i\}_{i=1}^{\infty}$ is the Fibonacci sequence with $F_0 = F_1 = 1$, then the g sequence can be written as $1, 1, F_1, F_2, F_3, F_4, F_7, F_8, \ldots$, which suggests that $g_{2k} = F_{2^k-1}$ and $g_{2k+1} = F_{2^k}$ for all positive integers k.

   We prove this by induction. Since the initial conditions (g_0 and g_1) of the sequence are the same as F_1 and F_2 respectively, the base case is complete. Therefore, assume that $g_{2n} = F_{2^n-1}$ and $g_{2n+1} = F_{2^n}$ and consider g_{2n+2} in addition to g_{2n+3}. The Fibonacci identities $F_{2k} = F_k^2 + F_{k-1}^2$ and $F_{2k-1} = 2F_kF_{k-1} - F_{k-1}^2$ solve the inductive step.

   For sake of completeness, we prove these identities in tandem by strong induction. For $F_{2k} = F_k^2 + F_{k-1}^2$, we can check that the base case $k = 1$ is consistent with this identity ($F_2 = 2 = 1 + 1 = F_1^2 + F_0^2$). For $F_{2k-1} = 2F_kF_{k-1} - F_{k-1}^2$, the base case of $k = 1$ is verified by $F_1 = 1 = 2(1) - 1 = 2F_1F_0 - F_0^2$, as desired. Therefore, we have completed the base case for induction. For the inductive hypothesis, assume that

$$F_{2k-2} = F_{k-1}^2 + F_{k-2}^2 \text{ and } F_{2k-3} = 2F_{k-1}F_{k-2} - F_{k-2}^2.$$

Note that

$$2F_kF_{k-1} - F_{k-1}^2 = 2(F_{k-1} + F_{k-2})F_{k-1} - F_{k-1}^2 = F_{k-1}^2 + 2F_{k-1}F_{k-2}$$
$$= (F_{k-1}^2 + F_{k-2}^2) + (2F_{k-1}F_{k-2} - F_{k-2}^2).$$

By this equation and the inductive hypothesis,

$$2F_kF_{k-1} - F_{k-1}^2 = (F_{k-1}^2 + F_{k-2}^2) + (2F_{k-1}F_{k-2} - F_{k-2}^2) = F_{2k-2} + F_{2k-3} = F_{2k-1},$$

as desired. Now, we need to finish the inductive step for the other equation:

$$F_k^2 + F_{k-1}^2 = (F_{k-1} + F_{k-2})^2 + F_{k-1}^2 = 2F_{k-1}^2 + 2F_{k-1}F_{k-2} + F_{k-2}^2$$
$$= 2(F_{k-1}^2 + F_{k-2}^2) + 2F_{k-1}F_{k-2} - F_{k-2}^2.$$

By the inductive hypothesis,

$$F_k^2 + F_{k-1}^2 = 2(F_{k-1}^2 + F_{k-2}^2) + 2F_{k-1}F_{k-2} - F_{k-2}^2 = 2F_{2k-2} + F_{2k-3}$$
$$= F_{2k-1} + F_{2k-2} = F_{2k},$$

as desired. Therefore, the induction is complete and both identities have been proven. (These identities may also be proven by Binet's formula or by matrix products.)

Now, consider the Fibonacci Sequence $\pmod 8$ and $\pmod{27}$. The first terms of the Fibonacci Sequence $\pmod 8$ are $1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, \ldots$ (repeat of the beginning) so the period of the Fibonacci Sequence $\pmod 8$ is 12. $2^k \equiv 2^{k+2} \pmod{12}$ for all $k \geq 2$, so $g_{2011} = F_{2^{1005}} \equiv F_8 \equiv 2 \pmod 8$. To calculate the period $\pmod{27}$, write out terms of the sequence $(1, 1, 2, 3, 5, 8, 13, 21, 7, 1, 8, 0, \ldots)$. Note that $F_{12} \equiv 0 \pmod{27}$. Therefore, for all indicies i such that $13 \leq i \leq 24$, $F_i \equiv 8F_{i-12} \pmod{27}$ since this part of the sequence is generated by $F_{12} \equiv 0$ and $F_{13} \equiv 8 \pmod{27}$. Therefore, $F_{23} \equiv 64 \equiv 10 \pmod{27}$. By the same logic as before, $F_i \equiv 10F_{i-24}$ if $25 \leq i \leq 36$. Therefore, $F_{35} \equiv 80 \equiv -1 \pmod{27}$. Therefore, for all i such that $36 \leq i \leq 71$, $F_i \equiv -F_{i-36} \pmod{27}$, so $F_{71} \equiv 1 \pmod{27}$. Since $F_{72} \equiv 0 \pmod{27}$, the Fibonacci sequence repeats after 72 terms. There is no smaller period, as that period would have to divide 36 or 24, even though we know that $F_{35} \equiv -1$ and $F_{23} \equiv 10 \pmod{27}$.

Now, $2^{1002} = 8^{334} \equiv (-1)^{334} \equiv 1 \pmod 9$, so $2^{1005} \equiv 8 \pmod{72}$. Therefore, $F_{2^{1005}} \equiv F_8 \pmod{27}$. Therefore, $g_{2011} \equiv F_8 \equiv 34 \pmod{216}$, so the answer is $\boxed{34}$.

8. Note that

$$\sum_{i=1}^{m} i^3 = \left(\sum_{i=1}^{m} i\right)^2$$

for all positive integers m. Therefore,

$$\sum_{i=k+1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{k(k+1)}{2}\right)^2.$$

The given equation therefore is the same as

$$\left(\frac{n(n+1)}{2}\right)^2 = (96^2(3))\left(\frac{k(k+1)}{2}\right)^2 + 48^2.$$

Since 48^2 divides the right side of this equation, $48|\frac{n(n+1)}{2}$, so we can write this equation as

$$\left(\frac{n(n+1)}{96}\right)^2 - 3\left(k(k+1)\right)^2 = 1.$$

This is Pell's equation. Therefore, $\frac{n(n+1)}{96}$ can be written as $\frac{(2+\sqrt{3})^m+(2-\sqrt{3})^m}{2}$ for any positive integer m. Furthermore, $k(k+1) = \frac{(2+\sqrt{3})^m-(2-\sqrt{3})^m}{2\sqrt{3}}$. Since $k \equiv 0, 3 \pmod 4$, $4|k(k+1)$ and $8|\frac{(2+\sqrt{3})^m-(2-\sqrt{3})^m}{\sqrt{3}}$. Note that

$$\frac{(2+\sqrt{3})^m-(2-\sqrt{3})^m}{\sqrt{3}} \equiv 2(3)^{\lfloor \frac{m}{2} \rfloor} \pmod 8$$

for odd m (since all terms besides the last term in the expansion are divisible by 8). This contradicts the fact that $k(k+1)$ is a multiple of 4, so m is even.

For even m, let $m = 2p$ for some integer p. Then

$$\frac{n(n+1)}{96} = \frac{(2+\sqrt{3})^m+(2-\sqrt{3})^m}{2}$$

is equivalent to

$$4n^2+4n = 192((2+\sqrt{3})^{2p}+(2-\sqrt{3})^{2p}) = 64\left(\sqrt{3}((2+\sqrt{3})^p-(2-\sqrt{3})^p)\right)^2 + 384.$$

Note that $\sqrt{3}((2+\sqrt{3})^p-(2-\sqrt{3})^p)$ is a positive integer, so WLOG call it x. Then $(2n+1)^2 = 64x^2+385$, so from difference of squares $(2n+1-8x)(2n+1+8x) = 5\cdot 7\cdot 11$. The only positive integer solutions to this equation are $(n,x) = (96, 24)$ and $(15, 3)$. $3 \neq \sqrt{3}((2+\sqrt{3})^p-(2-\sqrt{3})^p)$ for any p, but $p = 2$ yields the $x = 24$ case. Therefore, the only possible solution occurs when $n = 96$. Plugging this into the original equation shows that $k = 7$ so the answer is $96+7 = \boxed{103}$.