PUMaC 2011

Power Round: Geometry Revisited

> Stobaeus (one of Euclid's students): "But what shall I get by learning these things?"
> Euclid to his slave: "Give him three pence, since he must make gain out of what he learns."
>
> \- Euclid, Elements

1 Rules

These rules supersede any rules appearing elsewhere about the Power Test.

For each problem, you may use without proof any result (problem, lemma, proposition, exerice, or theorem) occurring earlier in the test, even if it's a problem your team has not solved. You may cite results from conjectures or subsequent problems only if your team solved them independently of the problem in which you wish to cite them. You may not cite parts of your proof of other problems: if you wish to use a lemma in multiple problems, reproduce it in each one.

It is not necessary to do the problems in order, although it is a good idea to read all the problems, so that you know what is permissible to assume when doing each problem. However, please collate the solutions in order in your solution packet. Each problem should start on a new page.

Using printed and noninteractive online references, computer programs, calculators, and Mathematica (or similar programs), is allowed. (If you find something online that you think trivializes part of the problem that wasn't already trivial, let us know—you won't lose points for it.) No communication with humans outside your team about the content of these problems is allowed.

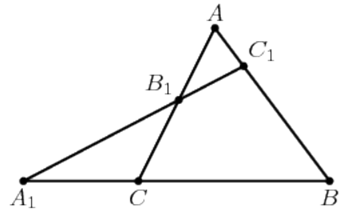
2 Prerequisites

We open with a quick exercise. When we say that $D \in BC$, we mean that D is a point anywhere in the line passing through BC (not necessarily just the line segment $\overline{BC}$ between them).

Exercise -9 (3 points). Let ABC be a triangle and let $D \in BC$, $E \in AC$, $F \in AB$, where the six points A, B, C, D, E, F are all distinct. Prove that if D, E, and F are collinear, then either none or exactly two of D, E, and F lie in their corresponding line segments

$\overline{BC}, \overline{AC}, \overline{AB}$, respectively.

Menelaus' theorem (triangle version).



Let ABC be a triangle and let $A_1 \in BC$, $B_1 \in AC$, $C_1 \in AB$, with none of A_1, B_1, and C_1 a vertex of ABC. Then A_1, B_1, C_1 are collinear if and only if

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1$$

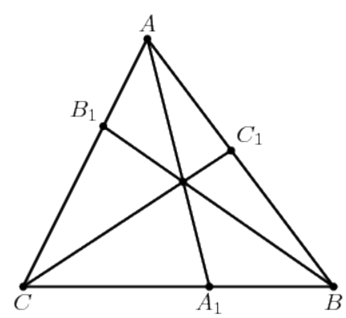
and exactly one or all three lie outside the segments $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. (We use the convention that all segments are unoriented throughout this test.)

Menelaus's Theorem (quadrilateral one-way version).
Let $A_1A_2A_3A_4$ be a quadrilateral and let d be a line which intersects the sides A_1A_2, A_2A_3, A_3A_4 and A_4A_1 at the points M_1, M_2, M_3, and M_4, respectively. Then,

$$\frac{M_1A_1}{M_1A_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \frac{M_3A_3}{M_3A_4} \cdot \frac{M_4A_4}{M_4A_1} = 1.$$

Exercise -8 (7 points). Prove the quadrilateral one-way version of Menelaus's Theorem.

Ceva's theorem.



Let ABC be a triangle and let $A_1 \in BC$, $B_1 \in AC$, $C_1 \in AB$, with none of A_1, B_1, and C_1 a vertex of ABC. Then AA_1, BB_1, CC_1 are concurrent if and only if

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1$$

and exactly one or all three lie on the segments $\overline{BC}$, $\overline{CA}$, $\overline{AB}$.

Exercise -7 (a) (1 point). In $\triangle ABC$, suppose D, E, and F are the midpoints of BC, CA, and AB, respectively. Prove that AD, BE, CF are concurrent.

(b) (2 points). In $\triangle ABC$, suppose AD is an angle bisector of $\angle BAC$, and similarly that BE, CF are angle bisectors. Prove that AD, BE, CF are concurrent.

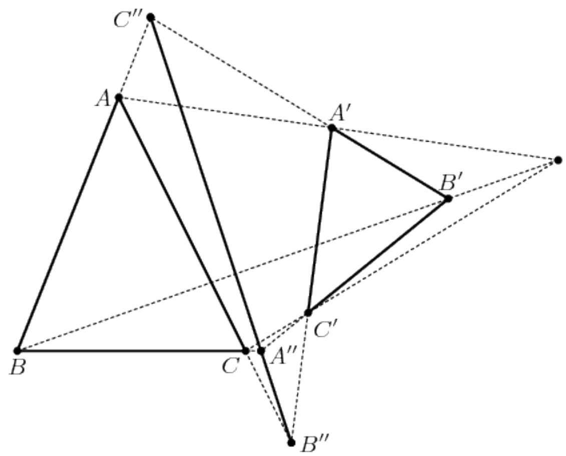
Exercise -6. Mass point geometry is a method useful in finding ratios of certain lengths in setups involving triangles and cevians. Imagine that X and Y are vertices of a triangle. We may assign masses $\mu(X), \mu(Y) > 0$ to the points X and Y. Then, the "center of mass" of the points X and Y is the point Z on XY such that $\mu(X) \cdot XZ = \mu(Y) \cdot YZ$. The problem-solving strategy is to choose the weights for the vertices of the triangles so that the centers of masses of the vertices of the triangle coincide with the important points in the diagram, such as endpoints of cevians. When the masses are chosen this way, then the center of mass of the triangle is the intersection of the cevians. For more details, examples, and rigorous definitions, one can refer to Wikipedia.

(a) (2 points). In $\triangle ABC$, D is in $\overline{BC}$, E is in $\overline{CA}$, and F is in $\overline{AB}$ such that AD, BE, CF concur at the point X, and $AF = 2, FB = 3, BD = 1, DC = 5$. Find AE/CE using mass points, and without using mass points.

(b) (2 points) Find AD/AX again using mass points, and without using mass points.

(c) (2 points). Find the ratio of the areas of $\triangle AFX$ to $\triangle CDX$ (using any method of your choice).

Desargues's theorem. Let ABC and $A'B'C'$ be two triangles. Let A'' be the intersection of BC and $B'C'$, B'' be the intersection of CA and $C'A'$, and C'' be the intersection of AB and $A'B'$, and assume all three of those intersections exist.



Then the lines AA', BB', and CC' are concurrent or all parallel if and only if A'', B'', and C'' are collinear.

Desargues's Theorem is true even if, for instance, BC and $B'C'$ are parallel, provided that one interprets the "intersection" of parallel lines appropriately. We won't state cases like the following separately in the future; you may want to google "line at infinity" if you're unfamiliar with them.

Exercise -5 (5 points). Let ABC and $A'B'C'$ be two triangles. Suppose that BC and $B'C'$ are parallel, CA and $C'A'$ are parallel, and AB and $A'B'$ are parallel. Prove that the lines AA', BB', and CC' are concurrent or all parallel.

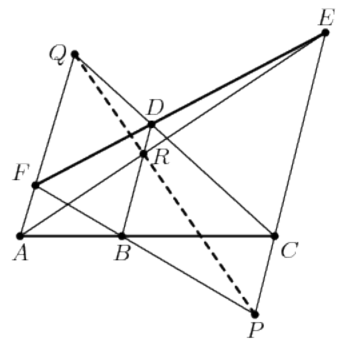
Pascal's theorem. Let A, B, C, D, E, F be six points all lying on the same circle. Then, the intersections of AB and DE, of BC and EF, and of CD and FA are collinear.

Keep an open mind for degenerate cases, where some of the points are equal! For this theorem, if two of the points are the same, then the line through them is the tangent to the circle at that point.

Exercise -4 (5 points). Suppose the incircle of $\triangle ABC$ is tangent to $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F, respectively. Let X, Y, Z be the intersections of AB and DE, BC and EF, CA and FD, respectively. Show that X, Y, Z are collinear.

In fact, we may replace the circle in Pascal's Theorem with an arbitrary conic section. When we consider the degenerate conic section given by two lines, we obtain the statement of Pappus's theorem:

Pappus's theorem. Given three collinear points A, B, C and three collinear points D, E, F, then if P is the intersection of BF and CE, Q of AF and CD, and R of AE and BD, then P, Q, R are also collinear.



Also, some projective geometry notions might come in handy at some point. We introduce below the notions of *harmonic division*, *harmonic pencil*, *pole*, *polar*, and mention some lemmas that you may want to be familiar with.

Let A, C, B, and D be four points on a line d. If they are in that order on d, the quadruplet $(ACBD)$ is called a harmonic division (or just "harmonic") if and only if

$$\frac{CA}{CB} = \frac{DA}{DB}.$$

Notice that this definition remains invariant if we swap the orders of A, B and/or C, D on the line. Thus, if the four points lie in the order A, D, B, C or B, C, A, D or B, D, A, C, we may define the harmonic division $(ACBD)$ in the same way.

Exercise -3 (3 points). Suppose $A = (1, 0)$, $B = (5, 4)$, and $C = (4, 3)$. Find D such that $(ACBD)$ is harmonic, E such that $(ACEB)$ is harmonic, and F such that $(AFCB)$ is harmonic.

If X is a point not lying on d, then the "pencil" $X(ACBD)$ (consisting of the four lines XA, XB, XC, XD) is called harmonic if and only if the quadruplet $(ACBD)$ is harmonic. So in this case, $X(ACBD)$ is usually called a *harmonic pencil*.

Now, the polar of a point P in the plane of a given circle Γ with center O is defined as the line containing all points Q such that the quadruplet $(PXQY)$ is harmonic, where X and Y are the intersection points with Γ of a line passing through P. This polar is actually a line perpendicular to OP; when P lies outside the circle, we can see from the following exercise that it is precisely the line determined by the tangency points of the circle with the tangents from P. We say that P is the pole of this line with respect to Γ.

Exercise -2 (2 points). Let O be the circle of radius 2 centered at the origin, and let $P = (a, a)$ for some $a \neq 0$. Find the polar of P with respect to O.

Given these concepts, there are four important lemmas that summarize the whole theory. Remember them by heart even though you might prefer not to use them in this test. They are quite beautiful.

Lemma 1. In a triangle ABC consider three points X, Y, Z on the sides BC, CA, and AB, respectively. If X' is the point of intersection of the line YZ with the extended side BC, then the quadruplet $(BXCX')$ is a harmonic division if and only if the lines AX, BY, CZ are concurrent.

Lemma 2. Let ℓ and ℓ' be two lines in plane and let P be a point in the same plane but not lying on either ℓ or ℓ'. Let A, B, C, D be points on ℓ and let A', B', C', D' be the intersections of the lines PA, PB, PC, PD with ℓ', respectively. Then

$$\frac{BA}{BC} : \frac{DA}{DC} = \frac{B'A'}{B'C'} : \frac{D'A'}{D'C'}.$$

Conversely, if $A = A'$; A, B, C, D and A', B', C', D' lie in those orders on ℓ and ℓ', respectively; and the cross-ratio above holds, then the lines BB', CC', DD' are concurrent.

Lemma 3. Let A, B, C, D be four points lying in this order on a line d. If X is a point not lying on this line, then if two of the following three propositions are true, then the third is also true:

- The quadruplet $(ABCD)$ is harmonic.
- XB is the internal angle bisector of $\angle AXC$.
- $XB \perp XD$.

Lemma 4. If P lies on the polar of Q with respect to some circle Γ, then Q lies on the polar of P.

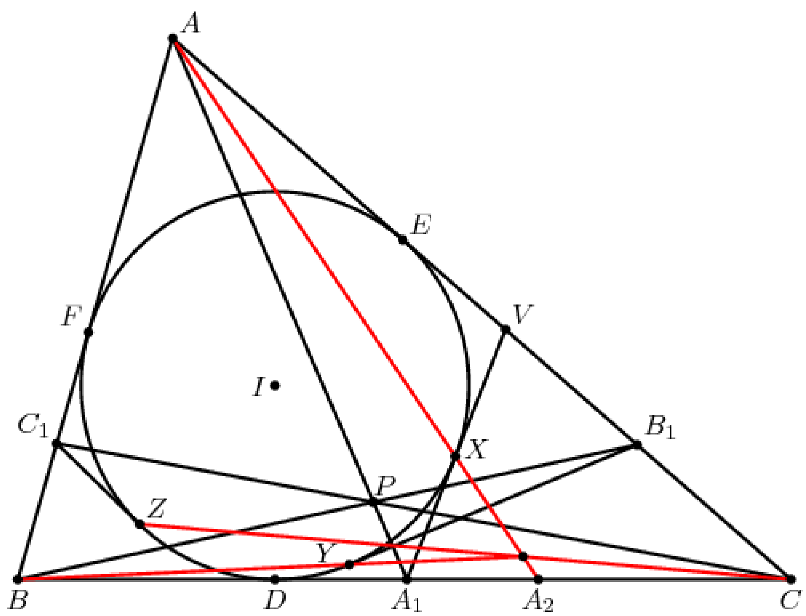
Exercise -1 (a) (3 points). Prove Lemma 1. (b) (6 points). Prove Lemma 2.

Reminder 0 (0 points). On every page you submit, put your team's name, a page count, and solutions to problems from at most one problem.

This is it. You are now ready to proceed the next section. As Erdős would advise you, keep your brain open!

3 The Test

General Setting. We are given a scalene triangle ABC in plane, with incircle Γ, and we let D, E, F be the tangency points of Γ with the sides BC, CA, and AB, respectively. Furthermore, let P be an arbitrary point in the *interior*[1] of the triangle ABC and let $A_1B_1C_1$ be its cevian triangle (that is, A_1, B_1, C_1 are the intersections of the lines AP, BP, CP with the sides BC, CA, and AB, respectively); from A_1, B_1, C_1 draw the tangents to Γ that are different from the triangle's sides BC, CA, AB, and let their tangency points with the incircle be X, Y, and Z, respectively. Now, with this picture in front of you, take a look at the following results! (ahem... and prove them!)



Proposition 1 (a) (6 points). Extend A_1X to intersect AC at V. Show that the lines AA_1, BV, and DE are concurrent.

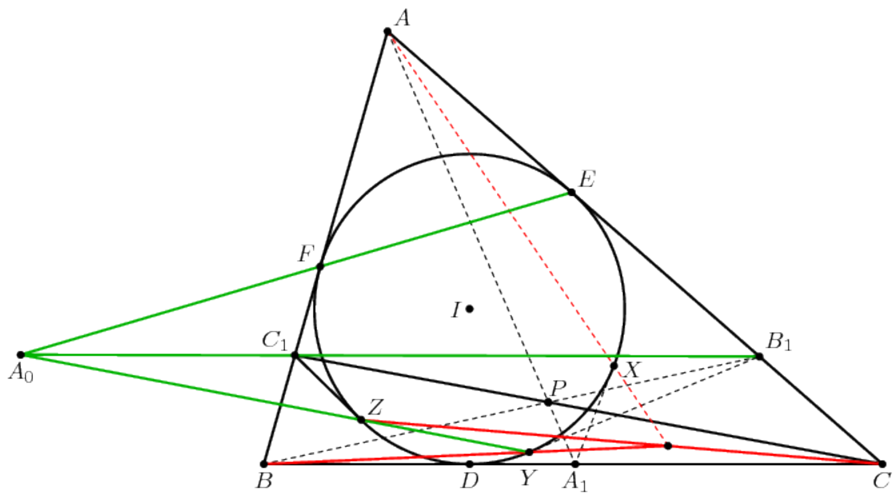
(b) (12 points). Let A_2 be the intersection point of AX with the sideline BC. Show that

$$\frac{A_2B}{A_2C} = \frac{s-c}{s-b} \cdot \frac{A_1B^2}{A_1C^2}.$$

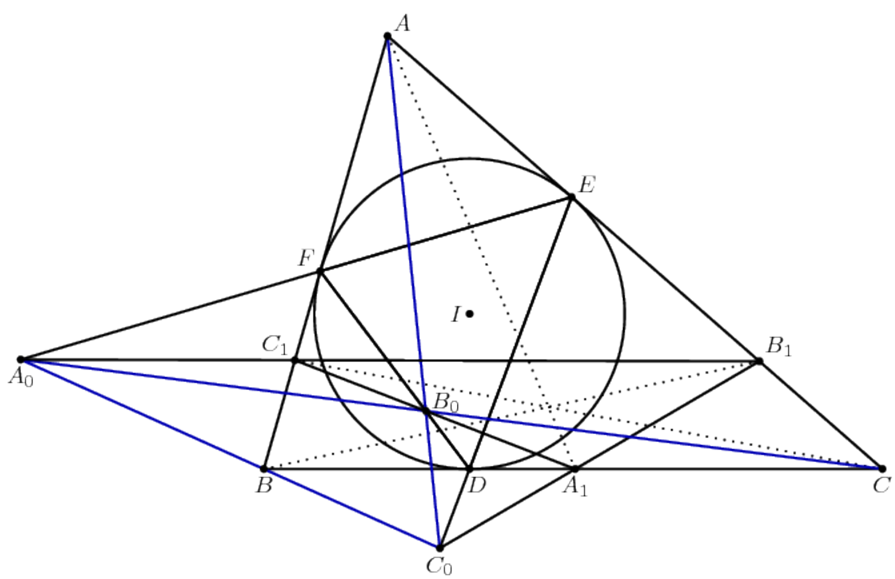
[Hint: Menelaus, Menelaus, Menelaus!]

(c) (2 point). Prove that the lines AX, BY, and CZ are concurrent.

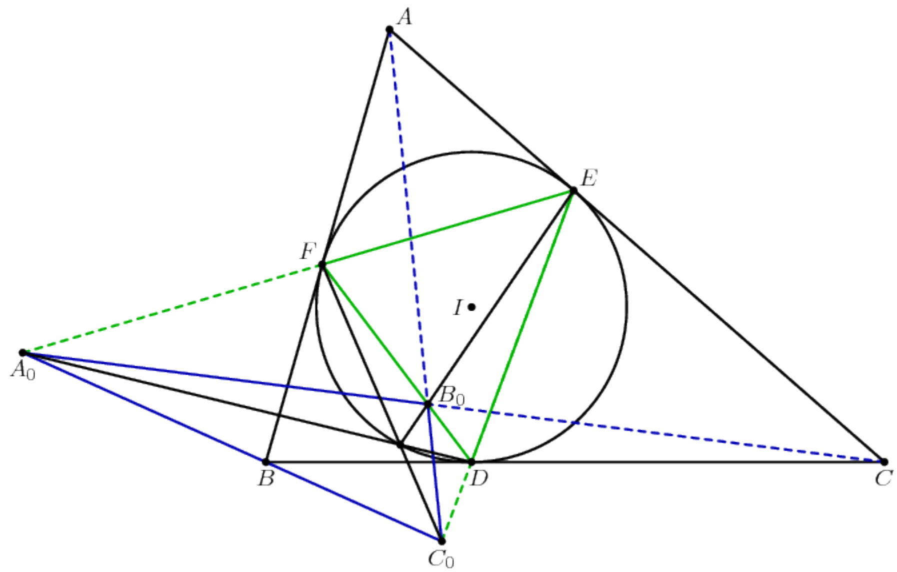
[1] The point P is considered inside ABC just for convenience, so please don't worry about the word *interior* that much; we just want nice, symmetrical drawings!)



Proposition 2 (14 points). The lines B_1C_1, EF, YZ concur at a point, say A_0. Similarly, the lines C_1A_1, FD, ZX and A_1B_1, DE, XY concur at points B_0 and C_0, respectively. [Hint: Pascal's theorem might be useful here.]



Proposition 3 (8 points). The points A_0, B, C_0 are collinear. Similarly, the points A_0, B_0, C are collinear, and A, B_0, C_0 are collinear. [Hint: Recall Lemma 2.]



Proposition 4 (14 points). The triangles $A_0B_0C_0$ and DEF are *perspective*[2] and their *perspector*[3] lies on the incircle of $\triangle ABC$.

Proposition 5 (a) (10 points). Let ABC be an arbitrary triangle, and A', B', C' be three points on BC, CA, AB. Let also A'', B'', C'' be three points on the sides $B'C'$, $C'A'$, $A'B'$ of triangle $A'B'C'$. Then consider the following three assertions:

(1) The lines AA', BB', CC' concur.

(2) The lines $A'A''$, $B'B''$, $C'C''$ concur.

(3) The lines AA'', BB'', CC'' concur.

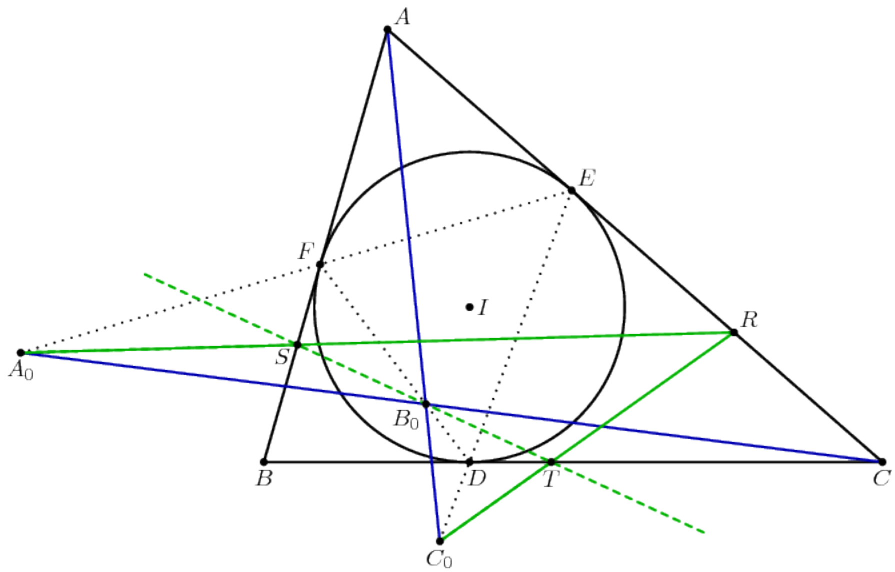
If any two of these three assertions are valid, then the third one must hold, too. [Hint: We didn't put the quadrilateral version of Menelaus in the Prerequisites section for nothing!]

(b) (4 points). The triangles $A_0B_0C_0$ and ABC are perspective.

Proposition 6 (10 points). The incenter I of ABC is the orthocenter of triangle $A_0B_0C_0$. [Hint: Lemma 2 might prove useful here.]

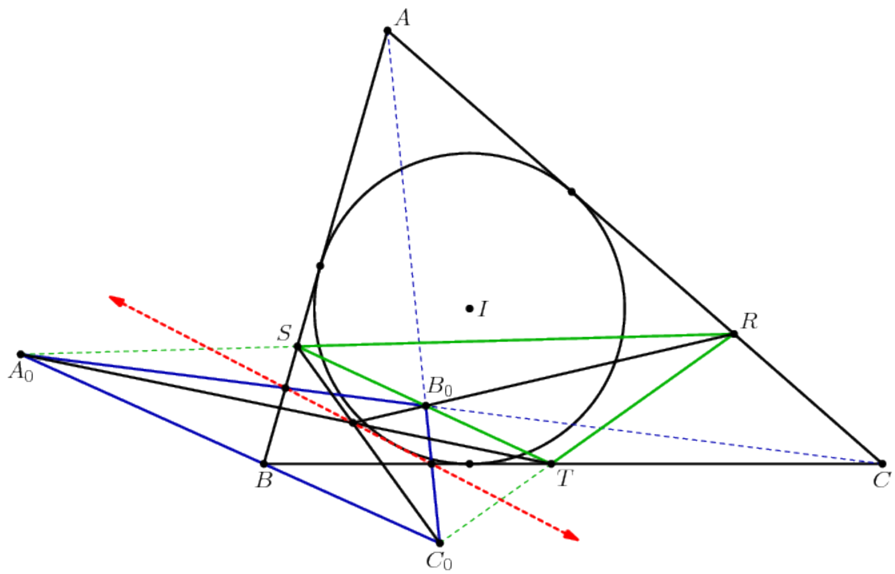
[2]Two triangles ABC and XYZ are said to be perspective if and only the lines AX, BY, CZ are concurrent.

[3]The perspector of two perspective triangles ABC and XYZ is precisely the common point of the lines AX, BY, CZ.



Proposition 7 (14 points). Let R a point on AC and consider S, T the intersections of RA_0, RC_0 with AB and BC respectively. Then, B_0 lies on ST.

Proposition 8 (14 points). The triangles ABC and TRS are perspective.



Proposition 9 (20 points). The triangles $A_0B_0C_0$ and TRS are perspective and the locus of the perspector as R varies along the line AC is a line tangent to the incircle of triangle ABC. [Hint: This is the capstone problem of the test, and its solution uses many of the results you've encountered throughout the test.]

4 Historical Fact/Challenge

When the point P chosen at the beginning of the test is the incenter, the Nagel point, or the orthocenter of triangle ABC, the configuration generates a "Feuerbach family" of circles. More precisely, maintaining the notations from the previous propositions, the circumcircle of triangle TRS always passes through the *Feuerbach point*[4] of triangle ABC, as R varies on the line AC. So, do try and have fun with it after finishing everything. However, you will *not* receive any credit for this problem.

[4] The incircle and the nine-point circle of a triangle ABC are tangent to each other and their tangency point is called the Feuerbach point of triangle ABC.