PUMaC 2011

Team Round Solutions

1 Across

- **1 Across.** Observe that the first digit of the 3-digit number GOT changes when we add three 2-digit numbers. Furthermore, one of the 2-digit numbers starts with a 2, and so we add at most $29 + 98 + 98 = 225$ to GOT. This means T is either 3 or 4 or 5. But then, TO is at most 59, and so in fact, we add at most $59 + 59 + 29 = 147$ to GOT. So in fact, T is either 3 or 4. So GOT is something of the form $2 * 3$ or $2 * 4$. Some calculator bashing now yields $(G, O, T, P) = (2, 6, 3, 1)$, so that $GOT \times TO = 9468$.

- **3 Across.** The only digits which mean something meaningful upside down are $1, 6, 8, 9, 0$. A prime can end with only 1 out of this list, so all strobogrammatic primes start and end with 1. Clearly, 11 is the first. Checking the three-digit numbers now give 101 and 181 as the next two, so $a = 181$. Enumerating the triangular numbers yields $1, 3, 6, 10, 15, 21, 28, 36$ etc. Clearly, $21 - 15$ and $21 + 15$ are both in the list, so $b = 15$. Therefore, $ab = 2715$.

- **6 Across.** This one is hard to solve, so leave it blank for now.

- **7 Across.** This is impossible to solve until you have all the solutions!

- **8 Across.** It is clear that $X_{2,2} = 4$ and $X_{2,3} = 8$. A little more checking yields $X_{2,4} = 512$, so we are looking for 4512.

- **9 Across.** You can possibly choose to leave this one blank as well, but if you are curious and have time, then you can figure it out with a bit of casework. Denoting x as $100x_1 + 10x_2 + x_3$ and similarly for y and z as well, observe that we know $\{x_i, y_i, z_i\}$ for $i \in \{1, 2, 3\}$ all represent different digits. From the first condition we get either $100x_1 + 10(x_2 - 9) + (x_3 - 3) = 100y_1 + 10y_2 + y_3$, $100(x_1 - 1) + 10x_2 + (x_3 + 7) = 100y_1 + 10y_2 + y_3$, or $100(x_1 - 1) + 10(x_2 - 1) + (x_3 - 3) = 100y_1 + 10y_2 + y_3$, and similarly for the second condition. Now, doing a bit of casework yields the unique solution $(819, 726, 543)$.

- **11 Across.** There are only so many pairs you have to check! Starting with the largest consecutive pair of primes $(89, 97)$, we easily see that the solution has to be $(47, 53)$.

- **12 Across.** This can be solved using induction, but it is hard, so it's better to leave it blank for now.

- **15 Across.** It is not too hard to see that x has to be 98. Checking Leyland numbers one by one, observe that the list goes like $8, 17, 32, 54, \ldots$, and so $y = 54$. Now, w is clearly a 3-digit number, and since it's a large 3-digit number which is still 3-digit when multiplied by 3, it follows that the first digit of w is 3 and the second digit is less than 4. Therefore, $2w$ and $3w$ start with 6 and 9 respectively, and so w cannot contain $6, 9, 5, 4, 8$ (the last few because of this clue). Therefore, the second digit of w must be 2 or 1. It's now routine to check that w must be 327. This leaves the digits 1 and 6 for z, and since it is a prime, it must be 61. Therefore, $(w, x, y, z) = (327, 98, 54, 61)$.

- **19 Across.** A brute force check in this case works. We get the second hoax number to be 58.

- **20 Across.** It is well known that $a = 16$ (in fact, it is the *only* integer expressible as both x^y and y^x for distinct x and y). For b, factor $1200 = 2^4 \cdot 3 \cdot 5^2$. A simple check gives you $39 = 24 + 10 + 5 = 25 + 8 + 6 = 20 + 15 + 4$. Finally, since c cannot contain the digits $5, 8, 1, 6, 3, 9$ (because of the previous clue), it follows that c is a 2-digit number using two digits from $\{2, 4, 7\}$. Since the whole thing is prime, c ends with 7. Checking 27 and 47, it is clear that only 27 is the sum of the digits of its cube 19683. So the 6-digit prime number must be 163927.

- **21 Across.** If $d \geq 4$, then d^7 already has 5 digits. Therefore, $d \in \{1, 2\}$. But obvious $d \neq 1$, because then abc can be at most $9^3 = 729$, which is not 4 digits. Therefore, $d = 2$ or $d = 3$. If $d = 3$, N is a multiple of 2187. There are only four 4-digit multiples, and it is easy to check that they do not satisfy the conditions. So $d = 2$. The multiples of 128 that end in 2 are the $4 \pmod 5$ multiples. A quick check now yields $6912 = 6 \cdot 9 \cdot 1 \cdot 2^7$.

- **22 Across.** If you have time, brute force this! If not, you can leave this clue blank.

2 Down

- **1 Down.** This can be solved with enough time by factoring 62604360 cleverly, and finding which combinations add to 252. If you try it that way, a good first step may be to figure out what the one-digit factor is. If all else fails, don't fill this in yet.

- **2 Down.** This is just a simultaneous equations problem! Denoting by G, A and C the original number of books, you get

$$\begin{aligned} G + A + C &= 2511, \\ 2G + A + C/2 &= 2919, \\ 2G + C/2 &= 2184. \end{aligned}$$

  Solving this system yields $(G, A, C) = (864, 735, 912)$.

- **3 Down.** Each face of $\mathscr{T}$ is an equilateral triangle of side 28, so its area is $\sqrt{3} \cdot 28^2/4 \approx 339.48$. Therefore, the total surface area of the whole figure is $339.48 * 8 \approx 2716$ square units. This gives the result.

- **4 Down.** Observe that $T_{25} - T_{24} = 25$. Therefore, the given sum is just $T_{24} + 1 + T_{24} + 2 + \ldots + T_{24} + 25 = 25 \cdot T_{24} + T_{25} = (24 \cdot 25 \cdot 25)/2 + (25 \cdot 26)/2 = 7500 + 325 = 7825$.

- **5 Down.** Even if s is hard to figure out, r is not; take all the multiples of 9 in order, and check the property. At 54, we get $54 = 3^2 + 3^2 + 6^2 = 7^2 + 2^2 + 1^2 = 5^2 + 5^2 + 2^2$. Therefore, s must be 63, so we are looking for 5463.

- **10 Down.** This is pretty hard, so leave it blank for now.

- **13 Down.** This is also fairly hard, so leave it blank.

- **14 Down.** The multiples of 17 hint gives it away. $(33, 34, 35)$ is clearly a cluster, as $33 = 3 \cdot 11, 34 = 2 \cdot 17, 35 = 5 \cdot 7$. So, $p + 1 = 34$. Now, $85, 86, 87$ is also a cluster, so $q = 85$. So we are looking for 8534.
- **16 Down.** We have $100a + 10b + c = (b + c)^a$. Since this is the least positive integer, we may as well try $a = 2$ ($a = 1$ would not work, because then the number of digits don't add up). If $200 + 10b + c = (b + c)^2$, then the square has to be either $225, 256$ or 289. Clearly the last one fits the bill, and so we are looking for 289.
- **17 Down.** These 2-element subsets will have intersections of size either 1 or 2 with $\mathfrak{Y}$. The number of ways to have a size-1 intersection is to pick something from $\mathfrak{Y}$ and something from $\mathfrak{X} - \mathfrak{Y}$, which is a total of $29 \cdot 3$. Similarly, the number of ways to have a size-2 intersection is to choose two things from $\mathfrak{Y}$, which is a total of $29 \cdot 28/2$. Therefore, we get 493 possible subsets.
- **18 Down.** This may be a bit tricky to check with a calculator, so leave this out for now!

At this stage, your crossword looks as follows. Note that we have assumed that you have not filled in the hard clues above, or even the ones which you could have afforded to skip.

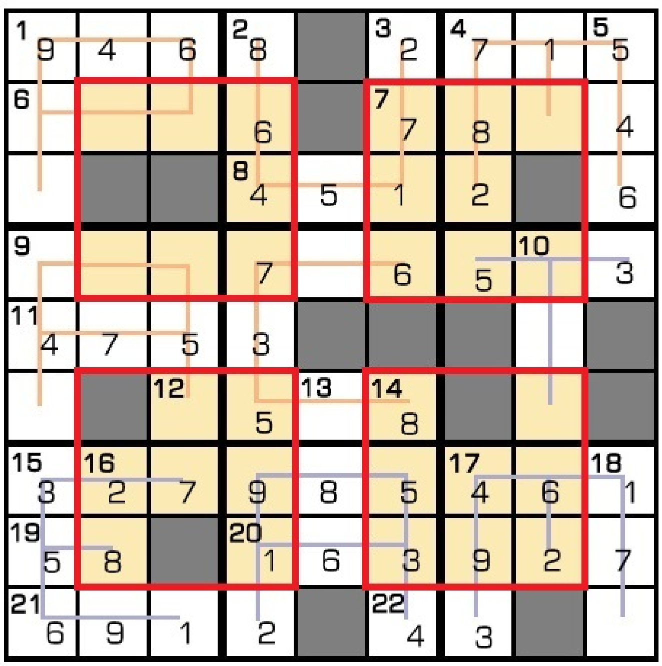

Figure 1: Halfway done!

After this, just use the Sudoku structure (keep the extra symmetries in mind!) to fill in the rest of the blanks to obtain the final solution.

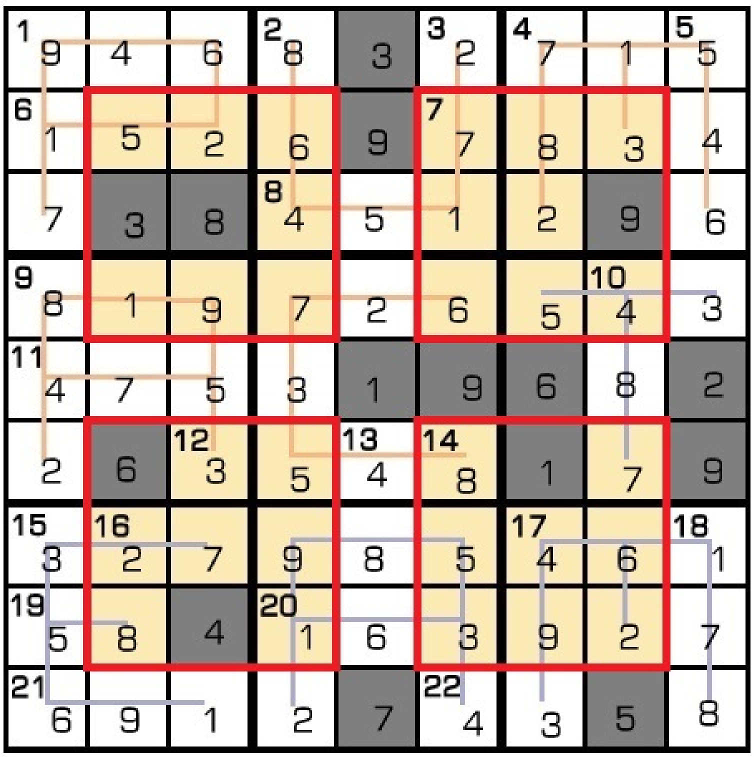

Figure 2: The complete solution