PUMaC 2011

Algebra B

1. [**3**] If we define $\otimes(a,b,c)$ by

$$\otimes(a,b,c) = \frac{\max(a,b,c) - \min(a,b,c)}{a+b+c-\min(a,b,c)-\max(a,b,c)},$$

compute $\otimes(\otimes(7,1,3), \otimes(-3,-4,2), 1)$.

2. [**3**] If a and b are the roots of $x^2 - 2x + 5$, what is $|a^8 + b^8|$?

3. [**4**] Let $f(x) = x^3 - 7x^2 + 16x - 10$. As x ranges over all integers, find the sum of distinct prime values taken on by $f(x)$.

4. [**4**] Let f be an invertible function defined on the complex numbers such that

$$z^2 = f(z + f(iz + f(-z + f(-iz + f(z + \ldots)))))$$

for all complex numbers z. Suppose $z_0 \neq 0$ satisfies $f(z_0) = z_0$. Find $1/z_0$. (Note: an invertible function is one that has an inverse).

5. [**5**] A polynomial p can be written as

$$p(x) = x^6 + 3x^5 - 3x^4 + ax^3 + bx^2 + cx + d.$$

Given that all roots of $p(x)$ are equal to either m or n where m and n are integers, compute $p(2)$.

6. [**6**] Shirley has a magical machine. If she inputs a positive even integer n, the machine will output $n/2$, but if she inputs a positive odd integer m, the machine will output $m + 3$. The machine keeps going by automatically using its output as a new input, stopping immediately before it obtains a number already processed. Shirley wants to create the longest possible output sequence possible with initial input at most 100. What number should she input?

7. [**7**] A sequence of real numbers $\{a_n\}_{n\geq 1}$ has the following property:

$$6a_n + 5a_{n-2} = 20 + 11a_{n-1} \text{ (for } n \geq 3).$$

The first two elements are $a_1 = 0, a_2 = 1$. Find the integer closest to a_{2011}.

8. [**8**] Let $\alpha_1, \alpha_2, \ldots, \alpha_6$ be a fixed labeling of the complex roots of $x^6 - 1$. Find the number of permutations $\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_6}\}$ of these roots such that if $P(\alpha_1, \ldots, \alpha_6) = 0$, then $P(\alpha_{i_1}, \ldots, \alpha_{i_6}) = 0$, where P is any polynomial with rational coefficients.