PUMaC 2011

Geometry B

1. [**3**] Let triangle ABC have $\angle A = 70^\circ, \angle B = 60^\circ$, and $\angle C = 50^\circ$. Extend altitude BH past H to point D so that $BD = BC$. Find $\angle BDA$ in degrees.

2. [**3**] Two logs of length 10 are laying on the ground touching each other. Their radii are 3 and 1, and the smaller log is fastened to the ground. The bigger log rolls over the smaller log without slipping, and stops as soon as it touches the ground again. The volume of the set of points swept out by the larger log as it rolls over the smaller one can be expressed as $n\pi$, where n is an integer. Find n.

3. [**4**] Let $ABCD$ be a trapezoid with AD parallel to BC, $AD = 2$, and $BC = 1$. Let M be the midpoint of AD, and let P be the intersection of BD with CM. Extend AP to meet segment CD at point Q. If the ratio $CQ/QD = a/b$, where a and b are positive integers and $\gcd(a, b) = 1$, find $a + b$.

4. [**4**] Let ω be a circle of radius 6 with center O. Let AB be a chord of ω having length 5. For any real constant c, consider the locus $\mathcal{L}(c)$ of all points P such that $PA^2 - PB^2 = c$. Find the largest value of c for which the intersection of $\mathcal{L}(c)$ and ω consists of just one point.

5. [**5**] Four circles are situated in the plane so that each is tangent to the other three. If three of the radii are 5, 5, and 8, the largest possible radius of the fourth circle is a/b, where a and b are positive integers and $\gcd(a, b) = 1$. Find $a + b$.

6. [**6**] A rectangular piece of paper has corners labeled A, B, C, and D, with $BC = 80$ and $CD = 120$. Let M be the midpoint of side AB. The corner labeled A is folded along line MD and the corner labeled B is folded along line MC until the segments AM and MB coincide. Let S denote the point in space where A and B meet. If H is the foot of the perpendicular from S to the original plane of the paper, find HM.

7. [**7**] Let ℓ_1 and ℓ_2 be two parallel lines, a distance of 15 apart. Points A and B lie on ℓ_1 while points C and D lie on ℓ_2 such that $\angle BAC = 30^\circ$ and $\angle ABD = 60^\circ$. The minimum value of $AD + BC$ is $a\sqrt{b}$, where a and b are positive integers and b is squarefree. Find $a + b$.

8. [**8**] Let ω_1 be a circle of radius 6, and let ω_2 be a circle of radius 5 that passes through the center O of ω_1. Let A and B be the points of intersection of the two circles, and let P be a point on major arc AB of ω_2. Let M and N be the second intersections of PA and PB with ω_1, respectively. Let S be the midpoint of MN. As P ranges over major arc AB of ω_2, the minimum length of segment SA is a/b, where a and b are positive integers and $\gcd(a, b) = 1$. Find $a + b$.