PUMaC 2011

Number Theory B

1. [**3**] What is the largest prime factor of 7999488?

2. [**3**] Two robots are programmed to communicate numbers using different bases. The first robot states: "I communicate in base 10, which interestingly is a perfect square. You communicate in base 16, which is not a perfect square." The second robot states: "I find it more interesting that the sum of our bases is the factorial of an integer." The second robot is referring to the factorial of which integer?

3. [**4**] The only prime factors of an integer n are 2 and 3. If the sum of the divisors of n (including itself) is 1815, find n.

4. [**4**] What is the largest positive integer $n < 1000$ for which there is a positive integer m satisfying
$$\text{lcm}(m, n) = 3m \times \gcd(m, n)?$$

5. [**5**] Find the sum of all primes p such that $7^p - 6^p + 2$ is divisible by 43.

6. [**6**] For how many ordered triplets of three positive integers is it true that their product is four more than twice their sum?

7. [**7**] Find the sum of all positive integers k with $k \leq 1000$ such that there exists an integer $n > k$ that satisfies
$$\sum_{i=k+1}^{n} i^3 = 3\left(\sum_{i=1}^{k} i\right)^2.$$

8. [**8**] Let $d(n)$ denote the number of divisors of n (including itself). You are given that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$
Find $p(6)$, where $p(x)$ is the unique polynomial with rational coefficients satisfying
$$p(\pi) = \sum_{n=1}^{\infty} \frac{d(n)}{n^2}.$$