PUMaC 2011

Number Theory B Solutions

1. Notice that 7999488 suspiciously close to 8000000. In fact, $7999488 = 8000000 - 512 = 200^3 - 2^9 = 2^9(25^3 - 1)$. We can either factor this as a difference of cubes, as $25^3 - 1 = (25-1)(25^2+25+1) = 24\cdot 651 = 2^3\cdot 3^2\cdot 7\cdot 31$, or as a difference of squares, since $25^3-1 = 5^6-1$. Therefore, $\boxed{31}$ is the largest prime factor.

2. Suppose that Robot 1's base is b_1 and Robot 2's base is b_2. From the first statement, we know that the first robot's base is $10_{b_1} = b_1$ and that the second robot's base is $b_2 = 16_{b_1} = b_1 + 6$. From the second statement, we know that $b_2 + b_1 = 2b_1 + 6 = n!$ for some integer n. Since b_1 is a perfect square, $b_1 \equiv 1$ or $4 \pmod 5$, so $n! \equiv 3$ or $4 \pmod 5$. However, this implies that $n \le 4$. Checking all $n \le 5$ shows that b_1 is a perfect square only when $n = 4$ ($b_1 = 9$), so $n = \boxed{4}$.

3. The sum of the divisors of $n = 2^i3^j$ is equal to $(1+2^1+2^2+\cdots+2^i)(1+3^1+3^2+\cdots+3^j) = 1815$, since each divisor of 2^i3^j is represented exactly once in the sum that results when the product is expanded. Let $A = 1+2^1+2^2+\cdots+2^i = 2^{i+1}-1$ and $B = 1+3^1+3^2+\cdots+3^j$, so that $AB = 1815 = 3\cdot 5\cdot 11^2$.

   Since $B \equiv 1 \pmod 3$, $3|A$. By Fermat's Little Theorem, $2^{i+1} - 1 \equiv 0 \pmod 3$ only when i is odd. For $i = 1$ we get $A = 3, B = 605$ which does not work. For $i = 3$ we get $A = 15, B = 121$, which holds for $n = \boxed{648}$. For $i = 5, 7, 9, 7|A, 17|A, 31|A$, respectively, and for $i > 10$, $A > 1815$.

4. Using the identity that $\operatorname{lcm}(m,n)\cdot\gcd(m,n) = m\cdot n$, it follows that

   $$3m \times \gcd(m,n) = \operatorname{lcm}(m,n) = \frac{m\cdot n}{\gcd(m,n)} \implies n = 3[\gcd(m,n)]^2.$$

   It follows that n must be three times a perfect square. If we set $m = \sqrt{n/3}$, which is an integer, it follows that

   $$\operatorname{lcm}(\sqrt{n/3}, n) = n = 3\sqrt{n/3}\cdot\sqrt{n/3} = 3\sqrt{n/3}\cdot\gcd(\sqrt{n/3}, n),$$

   as desired. Hence, every triple of a perfect square works as a value of n, and the largest such under 1000 is $3\cdot 18^2 = \boxed{972}$.

5. Note that $7^3 = 343 \equiv -1 \pmod{43}$ and that $6^6 = (6^3)^2 \equiv 1 \pmod{43}$. Therefore, for $p \equiv 0,1,2,3,4,5 \pmod 6$, $7^p - 6^p + 2 \equiv 2, 3, 15, 0, 32, 3 \pmod{43}$. Therefore, if $43|7^p - 6^p + 2$, $p \equiv 3 \pmod 6$. This means that $p = \boxed{3}$ is the only solution, so that the sum of all solutions is 3.

6. Let the set be $\{a, b, c\}$, and without loss of generality, suppose that $a \le b \le c$. Then

   $$abc - 2a - 2b - 2c = 4.$$

   If $c \ge b \ge a \ge 4$, then

   $$4 = abc - 2a - 2b - 2c \ge 16c - 2c - 2c - 2c = 10c \ge 40,$$

which is a contradiction. Thus, $a \in \{1, 2, 3\}$.

If $a = 1$, we get that $bc - 2b - 2c = 6$. Completing the rectangle, $bc - 2b - 2c + 4 = (b-2)(c-2) = 10$. Thus, $b - 2$ and $c - 2$ are a pair of (positive) factors of 10, and so must be equal to $1, 10$ or $2, 5$. Thus, we get the solutions $\{a, b, c\} = \{1, 3, 12\}$ and $\{1, 4, 7\}$, which can be permuted in 12 ways (6 for each solution).

If $a = 2$, we get that $2bc - 2b - 2c = 8$, so $bc - b - c + 1 = (b-1)(c-1) = 5$. Thus, $b - 1$ and $c - 1$ are a pair of factors of 5, and so must be equal to $1, 5$. This gives the solution $\{a, b, c\} = \{2, 2, 6\}$, which can be permuted in 3 ways.

If $a = 3$, we get that $3bc - 2b - 2c = 10$, so $9bc - 6b - 6c + 4 = (3b - 2)(3c - 2) = 34$. Then $3b - 2, 3c - 2$ are a pair of factors of 34 and $(3b - 2, 3c - 2) = (1, 34), (2, 17)$. In the first pair of solutions, we have $b = 1 \le a = 3$, and in the second pair of solutions, we do not get integer values for b, c.

In total, we have $\boxed{15}$ ordered triplets which satisfy the given conditions.

7. Rewrite $\sum_{i=k+1}^{n} i^3$ as $\sum_{i=1}^{n} i^3 - \sum_{i=1}^{k} i^3 = \frac{n^2(n+1)^2}{4} - \frac{k^2(k+1)^2}{4}$. Also, rewrite the right side as $3\frac{k^2(k+1)^2}{4}$. Therefore, $n^2(n+1)^2 = 4k^2(k+1)^2$. Taking the square root of both sides shows that $n(n+1) = 2k(k+1)$ (the negative case is not possible for positive n and k). Multiply both sides by 4 and add 1 to show that $(4n^2 + 4n + 1) = 2(4k^2 + 4k + 1) - 1$, which is equivalent to $-1 = (2n+1)^2 - 2(2k+1)^2$. Letting $x = 2k + 1$ and $y = 2n + 1$ shows that $-1 = y^2 - 2x^2$, with $(x, y) = (1, 1)$ as the base solution.

   All solutions can be obtained by computing $(1 + \sqrt{2})^{2j+1} = y + x\sqrt{2}$ for values of j such that $x \le 2001$. In this equation, $x = \binom{2j+1}{1} + \binom{2j+1}{3}2 + \ldots + \binom{2j+1}{2j+1}2^j$, which can be seen by expanding the expression for $y + x\sqrt{2}$ and equating coefficients of $\sqrt{2}$. Note that for $j = 4$, $x = 9 + 168 + 504 + 288 + 16 = 985$. For $j \ge 5$, $x \ge 11 + (11(5)(3))(2) + (11(2)(3)(7))(4) + (11(10)(3))8 + (11(5))16 + 32 > 2001$, so there are at most 4 solutions for $x > 1$ and $x \le 2001$. Note that $j = 1, 2,$ and 3 correspond to $x = 5, 29,$ and $7 + 70 + 84 + 8 = 169$. These correspond to $k = 2, 14, 84,$ and 492, so the desired sum is $\boxed{592}$.

8. First, show that this sum converges by showing that $d(n) \le 2\sqrt{n}$ for all $n \ge 1$. Let x be some divisor of n with $x \le \sqrt{n}$. There is a one to one correspondence between divisors at most $\sqrt{n}$ and divisors at least $\sqrt{n}$ (map x to $\frac{n}{x}$ for any $x \le n$). However, there are at most $\sqrt{n}$ divisors of n that are at most $\sqrt{n}$, so $d(n) \le 2\sqrt{n}$ for all positive integers n. Therefore, the given sum is bounded above by

$$\sum_{n=1}^{\infty} \frac{2}{x^{\frac{3}{2}}}$$

   which converges.

   First solution: Note that $d(n) = \sum_{k|n} 1$, so $\frac{d(n)}{n^2} = \sum_{k|n} \frac{1}{k^2(n/k)^2}$. This means that the desired sum is equal to

$$\sum_{n=1}^{\infty} \sum_{k|n} \frac{1}{k^2(n/k)^2} = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{k^2m^2}.$$

This can be seen by letting $m = \frac{n}{k}$ and switching the order of summation (which is valid by the absolute convergence of the original series). Therefore, we can write this series as

$$\left(\sum_{m=1}^{\infty} \frac{1}{m^2} \right)^2 = \frac{\pi^4}{36}.$$

Hence $p(x) = \frac{x^4}{36}$ and $p(6) = \boxed{36}$. $p(x)$ is unique because π is transcendental.

Second solution: Since the original sum converges, one may use the unique prime factorization of the integers and the fact that $d(n)$ is multiplicative to factor the given sum as

$$\prod_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{d(p_i^j)}{p_i^{2j}}$$

where p_i is the ith prime. Note also that $d(p^j) = j + 1$ for any prime p. Now, compute the sum

$$S(x) = \sum_{j=0}^{\infty} (j + 1)x^j$$

To compute a closed form for $S(x)$, consider $xS(x) = \sum_{j=1}^{\infty} jx^j$ and $S(x) - xS(x) = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ for $|x| < 1$. Therefore, $S(x) = \frac{1}{(1-x)^2} = \left(\sum_{j=0}^{\infty} x^j \right)^2$. Letting $x = \frac{1}{p_i}$ for each term of the infinite product shows that the desired sum is equal to $(\sum_{j=1}^{\infty} \frac{1}{j^2})^2 = \frac{\pi^4}{36}$. Hence $p(x) = \frac{x^4}{36}$ and $p(6) = \boxed{36}$.