1. Notice that 7999488 suspiciously close to 8000000. In fact, 7999488 = 8000000 − 512 = 200^3 − 2^9 = 2^9(25^3 − 1). We can either factor this as a difference of cubes, as 25^3 − 1 = (25−1)(25^2+25+1) = 24·651 = 2^3·3^2·7·31, or as a difference of squares, since 25^3−1 = 5^6−1. Therefore, 31 is the largest prime factor.

2. Suppose that Robot 1’s base is b_1 and Robot 2’s base is b_2. From the first statement, we know that the first robot’s base is 10b_1 = b_1 and that the second robot’s base is b_2 = 16b_1 = b_1 + 6. From the second statement, we know that b_2 + b_1 = 2b_1 + 6 = n! for some integer n. Since b_1 is a perfect square, b_1 ≡ 1 or 4 (mod 5), so n! ≡ 3 or 4 (mod 5). However, this implies that n ≤ 4. Checking all n ≤ 5 shows that b_1 is a perfect square only when n = 4 (b_1 = 9), so n = 4.

3. The sum of the divisors of n = 2^i·3^j is equal to (1+2^1+2^2+⋯+2^i)(1+3^1+3^2+⋯+3^j) = 1815, since each divisor of 2^i·3^j is represented exactly once in the sum that results when the product is expanded. Let A = 1+2^1+2^2+⋯+2^i = 2^{i+1}−1 and B = 1+3^1+3^2+⋯+3^j, so that AB = 1815 = 3·5·11^2. Since B ≡ 1 (mod 3), 3|A. By Fermat’s Little Theorem, 2^{i+1}−1 ≡ 0 (mod 3) only when i is odd. For i = 1 we get A = 3, B = 605 which does not work. For i = 3 we get A = 15, B = 121, which holds for n = 648. For i = 5, 7, 9, 7|A, 17|A, 31|A, respectively, and for i > 10, A > 1815.

4. Using the identity that lcm (m, n) · gcd (m, n) = m · n, it follows that

\[ 3m \times \gcd (m, n) = \text{lcm} (m, n) = \frac{m \cdot n}{\gcd (m, n)} \implies n = 3[\gcd (m, n)]^2. \]

It follows that n must be three times a perfect square. If we set m = \sqrt{n/3}, which is an integer, it follows that

\[ \text{lcm} \left( \sqrt{n/3}, n \right) = n = 3\sqrt{n/3} \cdot \sqrt{n/3} = 3\sqrt{n/3} \cdot \gcd \left( \sqrt{n/3}, n \right) \]

as desired. Hence, every triple of a perfect square works as a value of n, and the largest such under 1000 is \(3 \cdot 18^2 = 972\).

5. Note that 7^3 = 343 ≡ −1 (mod 43) and that 6^6 = (6^3)^2 ≡ 1 (mod 43). Therefore, for \(p \equiv 0, 1, 2, 3, 4, 5 \pmod{6}\), \(7^p - 6^p + 2 \equiv 2, 3, 15, 0, 32, 3 \pmod{43}\). Therefore, if 43|7^p - 6^p + 2, \(p \equiv 3 \pmod{6}\). This means that \(p = 3\) is the only solution, so that the sum of all solutions is 3.

6. Let the set be \(\{a, b, c\}\), and without loss of generality, suppose that \(a \leq b \leq c\). Then

\[ abc - 2a - 2b - 2c = 4. \]

If \(c \geq b \geq a \geq 4\), then

\[ 4 = abc - 2a - 2b - 2c \geq 16c - 2c - 2c - 2c = 10c \geq 40, \]

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which is a contradiction. Thus, \( a \in \{1, 2, 3\} \).

If \( a = 1 \), we get that \( bc - 2b - 2c = 6 \). Completing the rectangle, \( bc - 2b - 2c + 4 = (b-2)(c-2) = 10 \). Thus, \( b-2 \) and \( c-2 \) are a pair of (positive) factors of 10, and so must be equal to 1,10 or 2,5. Thus, we get the solutions \( \{a,b,c\} = \{1,3,12\} \) and \( \{1,4,7\} \), which can be permuted in 12 ways (6 for each solution).

If \( a = 2 \), we get that \( 2bc - 2b - 2c = 8 \), so \( bc - b - c + 1 = (b-1)(c-1) = 5 \). Thus, \( b - 1 \) and \( c - 1 \) are a pair of factors of 5, and so must be equal to 1,5. This gives the solution \( \{a,b,c\} = \{2,2,6\} \), which can be permuted in 3 ways.

If \( a = 3 \), we get that \( 3bc - 2b - 2c = 10 \), so \( 9bc - 6b - 6c + 4 = (3b-2)(3c-2) = 34 \). Then \( 3b - 2, 3c - 2 \) are a pair of factors of 34 and \( (3b - 2, 3c - 2) = (1,34), (2,17) \). In the first pair of solutions, we have \( b = 1 \leq a = 3 \), and in the second pair of solutions, we do not get integer values for \( b,c \).

In total, we have 15 ordered triplets which satisfy the given conditions.

7. Rewrite \( \sum_{i=k+1}^{n} i^3 - \sum_{i=1}^{n} i^3 = \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} - \frac{k^2(k+1)^2}{4} \). Also, rewrite the right side as \( 3\frac{k^2(k+1)^2}{4} \).

Therefore, \( n^2(n+1)^2 = 4k^2(k+1)^2 \). Taking the square root of both sides shows that \( n(n+1) = 2k(k+1) \) (the negative case is not possible for positive \( n \) and \( k \)). Multiply both sides by 4 and add 1 to show that \( 4n^2 + 4n + 1 = 2(4k^2 + 4k + 1) - 1 \), which is equivalent to \( -1 = (2n + 1)^2 - 2(2k + 1)^2 \). Letting \( x = 2k + 1 \) and \( y = 2n + 1 \) shows that \( -1 = y^2 - 2x^2 \), with \( (x,y) = (1,1) \) as the base solution.

All solutions can be obtained by computing \( (1 + \sqrt{2})^{2j+1} = y + x\sqrt{2} \) for values of \( j \) such that \( x \leq 2001 \). In this equation, \( x = \left(\frac{2j+1}{1}\right)^1 + \left(\frac{2j+1}{3}\right)^2 + \ldots + \left(\frac{2j+1}{2j+1}\right)^{2j} \), which can be seen by expanding the expression for \( y + x\sqrt{2} \) and equating coefficients of \( \sqrt{2} \). Note that for \( j = 4 \), \( x = 9 + 168 + 504 + 288 + 16 = 985 \). For \( j \geq 5 \), \( x \geq 11 + (11(5)(3))(2) + (11(2)(3)(7))(4) + (11(10)(3))8 + (11(5))16 + 32 > 2001 \), so there are at most 4 solutions for \( x > 1 \) and \( x \leq 2001 \). Note that \( j = 1, 2, 3 \) correspond to \( x = 5, 29, 7 + 70 + 84 + 8 = 169 \). These correspond to \( k = 2, 14, 84, \) and \( 492 \), so the desired sum is \( \frac{592}{2} \).

8. First, show that this sum converges by showing that \( d(n) \leq 2\sqrt{n} \) for all \( n \geq 1 \). Let \( x \) be some divisor of \( n \) with \( x \leq \sqrt{n} \). There is a one to one correspondence between divisors at most \( \sqrt{n} \) and divisors at least \( \sqrt{n} \) (map \( x \) to \( \frac{n}{x} \) for any \( x \leq n \)). However, there are at most \( \sqrt{n} \) divisors of \( n \) that are at most \( \sqrt{n} \), so \( d(n) \leq 2\sqrt{n} \) for all positive integers \( n \). Therefore, the given sum is bounded above by

\[
\sum_{n=1}^{\infty} \frac{2}{x^2}
\]

which converges.

**First solution**: Note that \( d(n) = \sum_{k|n} 1 \), so \( \frac{d(n)}{n^2} = \sum_{k|n} \frac{1}{k^2(n/k)^2} \). This means that the desired
sum is equal to
\[ \sum_{n=1}^{\infty} \sum_{k|n} \frac{1}{k^2(n/k)^2} = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{k^2m^2}. \]

This can be seen by letting \( m = \frac{n}{k} \) and switching the order of summation (which is valid by the absolute convergence of the original series). Therefore, we can write this series as
\[ \left( \sum_{m=1}^{\infty} \frac{1}{m^2} \right)^2 = \frac{\pi^4}{36}. \]

Hence \( p(x) = \frac{x^4}{36} \) and \( p(6) = 36 \). \( p(x) \) is unique because \( \pi \) is transcendental.

**Second solution:** Since the original sum converges, one may use the unique prime factorization of the integers and the fact that \( d(n) \) is multiplicative to factor the given sum as
\[ \prod_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{d(p_i^j)}{p_i^{2j}} \]
where \( p_i \) is the \( i \)th prime. Note also that \( d(p^j) = j + 1 \) for any prime \( p \). Now, compute the sum
\[ S(x) = \sum_{j=0}^{\infty} (j + 1)x^j \]
To compute a closed form for \( S(x) \), consider \( xS(x) = \sum_{j=1}^{\infty} jx^j \) and \( S(x) - xS(x) = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x} \) for \( |x| < 1 \). Therefore, \( S(x) = \frac{1}{(1-x)^2} = \left( \sum_{j=0}^{\infty} x^j \right)^2 \). Letting \( x = \frac{1}{p_i} \) for each term of the infinite product shows that the desired sum is equal to \( (\sum_{j=1}^{\infty} \frac{1}{j})^2 = \frac{\pi^4}{36} \). Hence \( p(x) = \frac{x^4}{36} \) and \( p(6) = [36] \).