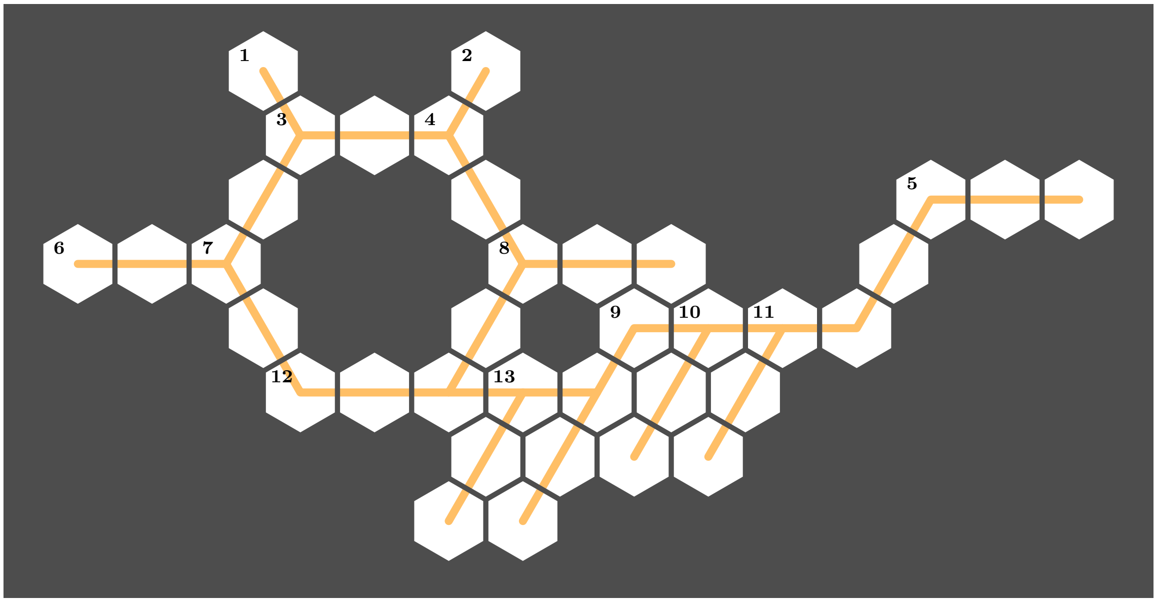
PUMaC 2012

Team Round

1 Instructions



- **Time limit: 20 minutes.**
- Fill in the crossword above with answers to the problems below.
- Notice that there are three directions instead of two. You are probably used to "down" and "across," but this crossword has "1," "$e^{4\pi i/3}$," and "$e^{5\pi i/3}$." You can think of these labels as complex numbers pointing in the direction to fill in the spaces. In other words "1" means "across," "$e^{4\pi i/3}$" means "down and to the left," and "$e^{5\pi i/3}$" means "down and to the right."
- To fill in the answer to, for example, 12 across, start at the hexagon labeled 12, and write the digits, proceeding to the right along the gray line. (Note: 12 across has space for exactly 5 digits.)
- Each hexagon is worth one point, and must be filled by something from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Note that π is not in the set, and neither is i, nor $\sqrt{2}$, nor ♡, etc.
- None of the answers will begin with a 0.
- "Concatenate a and b" means to write the digits of a, followed by the digits of b. For example, concatenating 10 and 3 gives 103. (It's not the same as concatenating 3 and 10.)
- Calculators are allowed!
- THIS SHEET IS PROVIDED FOR YOUR REFERENCE ONLY. DO NOT TURN IN THIS SHEET. TURN IN THE OFFICIAL ANSWER SHEET PROVIDED TO THE TEAM. OTHERWISE YOU WILL GET A SCORE OF ZERO! ZERO! ZERO! AND WHILE SOMETIMES "!" MEANS FACTORIAL, IN THIS CASE IT DOES NOT.
- Good luck, and have fun!

2 The problems

2.1 Across (1)

3. (3 digits) Suppose you draw 5 vertices of a convex pentagon (but not the sides!). Let N be the number of ways you can draw at least 0 straight line segments between the vertices so that no two line segments intersect in the interior of the pentagon. What is $N - 64$? (Note what the question is asking for! You have been warned!)

5. (3 digits) Among integers $\{1, 2, \ldots, 10^{2012}\}$, let n be the number of numbers for which the sum of the digits is divisible by 5. What are the first three digits (from the left) of n?

6. (3 digits) Bob is punished by his math teacher and has to write all perfect squares, one after another. His teacher's blackboard has space for exactly 2012 digits. He can stop when he cannot fit the next perfect square on the board. (At the end, there might be some space left on the board - he does not write only part of the next perfect square.) If n^2 is the largest perfect square he writes, what is n?

8. (3 digits) How many positive integers n are there such that $n \leq 2012$, and the greatest common divisor of n and 2012 is a prime number?

9. (4 digits) I have a random number machine generator that is very good at generating integers between 1 and 256, inclusive, with equal probability. However, right now, I want to produce a random number between 1 and n, inclusive, so I do the following:

   - I use my machine to generate a number between 1 and 256. Call this a.
   - I take a and divide it by n to get remainder r. If $r \neq 0$, then I record r as the randomly generated number. If $r = 0$, then I record n instead.

   Note that this process does not necessarily produce all numbers with equal probability, but that is okay. I apply this process twice to generate two numbers randomly between 1 and 10. Let p be the probability that the two numbers are equal. What is $p \cdot 2^{16}$?

12. (5 digits) You and your friend play the following dangerous game. You two start off at some point (x, y) on the plane, where x and y are nonnegative integers.

    When it is player A's turn, A tells his opponent B to move to another point on the plane. Then A waits for a while. If B is not eaten by a tiger, then A moves to that point as well.

    From a point (x, y) there are three places A can tell B to walk to: leftwards to $(x-1, y)$, downwards to $(x, y-1)$, and simultaneously downwards and leftwards to $(x-1, y-1)$. However, you cannot move to a point with a negative coordinate.

    Now, what was this about being eaten by a tiger? There is a tiger at the origin, which will eat the first person that goes there! Needless to say, you lose if you are eaten.

    Consider all possible starting points (x, y) with $0 \leq x \leq 346$ and $0 \leq y \leq 346$, and x and y are not both zero. Also suppose that you two play strategically, and you go first (i.e., by telling your friend where to go). For how many of the starting points do you win?

2.2 Down and to the left ($e^{4\pi i/3}$)

2. (2 digits) $ABCDE$ is a pentagon with $AB = BC = CD = \sqrt{2}$, $\angle ABC = \angle BCD = 120$ degrees, and $\angle BAE = \angle CDE = 105$ degrees. Find the area of triangle $\triangle BDE$. Your answer in its simplest form can be written as $\frac{a+\sqrt{b}}{c}$, where where a, b, c are integers and b is square-free. Find abc.

3. (3 digits) Suppose x and y are integers which satisfy

$$\frac{4x^2}{y^2} + \frac{25y^2}{x^2} = \frac{10055}{x^2} + \frac{4022}{y^2} + \frac{2012}{x^2y^2} - 20$$

What is the maximum possible value of $xy - 1$?

5. (3 digits) Find the area of the set of all points in the plane such that there exists a square centered around the point and having the following properties:

- The square has side length $7\sqrt{2}$.
- The boundary of the square intersects the graph of $xy = 0$ at at least 3 points.

8. (3 digits) Princeton Tiger has a mom that likes yelling out math problems. One day, the following exchange between Princeton and his mom occurred:

- Mom: Tell me the number of zeros at the end of 2012!
- PT: Huh? 2012 ends in 2, so there aren't any zeros.
- Mom: No, the exclamation point at the end was not to signify me yelling. I was not asking about 2012, I was asking about 2012!.

What is the correct answer?

9. (4 digits) Define the following:

- $A = \sum_{n=1}^{\infty} \frac{1}{n^6}$
- $B = \sum_{n=1}^{\infty} \frac{1}{n^6+1}$
- $C = \sum_{n=1}^{\infty} \frac{1}{(n+1)^6}$
- $D = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$
- $E = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^6}$

Consider the ratios $\frac{B}{A}, \frac{C}{A}, \frac{D}{A}, \frac{E}{A}$. Exactly one of the four is a rational number. Let that number be r/s, where r and s are nonnegative integers and $\gcd(r, s) = 1$. Concatenate r, s.

(It might be helpful to know that $A = \frac{\pi^6}{945}$.)

10. (3 digits) You have a sheet of paper, which you lay on the xy plane so that its vertices are at $(-1,0),(1,0),(1,100),(-1,100)$. You remove a section of the bottom of the paper by cutting along the function $y=f(x)$, where f satisfies $f(1)=f(-1)=0$. (In other words, you keep the bottom two vertices.)

    You do this again with another sheet of paper. Then you roll both of them into identical cylinders, and you realize that you can attach them to form an L-shaped elbow tube.

    We can write $f(\frac{1}{3})+f(\frac{1}{6})=\frac{a+\sqrt{b}}{\pi c}$, where a,b,c are integers and b is square-free. Find $a+b+c$.

11. (3 digits) Let

$$\Xi(x)=2012(x-2)^2+278(x-2)\sqrt{2012+e^{x^2-4x+4}}+139^2+(x^2-4x+4)e^{x^2-4x+4}$$

find the area of the region in the xy-plane satisfying:

$$\{x\geq 0 \quad \text{and} \quad x\leq 4 \quad \text{and} \quad y\geq 0 \quad \text{and} \quad y\leq \sqrt{\Xi(x)}\}$$

13. (3 digits) Three cones have bases on the same plane, externally tangent to each other. The cones all face the same direction. Two of the cones have radii of 2, and the other cone has a radius of 3. The two cones with radii 2 have height 4, and the other cone has height 6. Let V be the volume of the tetrahedron with three of its vertices as the three vertices of the cones and the fourth vertex as the center of the base of the cone with height 6. Find V^2.

2.3 Down and to the right ($e^{5\pi i/3}$)

1. (2 digits) For some reason, people in math problems like to paint houses. Alice can paint a house in one hour. Bob can paint a house in six hours. If they work together, it takes them seven hours to paint a house. You might be thinking "What? That's not right!" but I did not make a mistake.

    When Alice and Bob work together, they get distracted very easily and simultaneously send text messages to each other. When they are texting, they are not getting any work done. When they are not texting, they are painting at their normal speeds (as if they were working alone). Carl, the owner of the house decides to check up on their work. He randomly picks a time during the seven hours. The probability that they are texting during that time can be written as r/s, where r and s are integers and $\gcd(r,s)=1$. What is $r+s$?

4. (3 digits) Let $a_1=2+\sqrt{2}$ and $b_1=\sqrt{2}$, and for $n\geq 1$, $a_{n+1}=|a_n-b_n|$ and $b_{n+1}=a_n+b_n$. The minimum value of $\frac{a_n^2+a_nb_n-6b_n^2}{6b_n^2-a_n^2}$ can be written in the form $a\sqrt{b}-c$, where a,b,c are integers and b is square-free. Concatenate c,b,a (in that order!).

7. (3 digits) How many solutions are there to $a^{503}+b^{1006}=c^{2012}$, where a,b,c are integers and $|a|,|b|,|c|$ are all less than 2012?

– Alan Chang