PUMaC 2012

Algebra A Solutions

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1. [**3**] Compute the smallest positive integer a for which $\sqrt{a+\sqrt{a+\ldots}} - \dfrac{1}{a+\frac{1}{a+..}} > 7$.

   Solution: Let the first term be x and the second y. Then we have $x = \sqrt{a+x}$ and $y = \frac{1}{a+y}$ After solving this system in terms of a, we can write the difference as

   $$\frac{1}{2}(a+1+\sqrt{4a+1}-\sqrt{a^2+4})$$

   Further simplification gives:

   $$a-\sqrt{a^2+4}+\sqrt{4a+1} > 13,$$

   where the first term is a very small negative number. Thus $a = 42$ does not quite get us there but $a = 43$ does.

   Answer: $\boxed{43}$

2. [**3**] If x, y, and z are real numbers with $\dfrac{x-y}{z} + \dfrac{y-z}{x} + \dfrac{z-x}{y} = 36$, find

   $$2012 + \frac{x-y}{z}\cdot\frac{y-z}{x}\cdot\frac{z-x}{y}$$

   Solution:

   $$\begin{aligned}
36 &= \frac{(x-y)xy+(y-z)yz+(z-x)xz}{xyz} \\
&= \frac{(x-y)xy+y^2z-yz^2+xz^2-x^2z}{xyz} \\
&= \frac{(x-y)xy-(x+y)(x-y)z+z^2(x-y)}{xyz} \\
&= \frac{(x-y)(xy-xz-yz+z^2)}{xyz} \\
&= \frac{(x-y)(y-z)(x-z)}{xyz} \\
36 &= -\frac{(x-y)(y-z)(z-x)}{xyz} \\
2012 + \frac{(x-y)(y-z)(z-x)}{xyz} &= 2012-36 = 1976
\end{aligned}$$

   Answer: $\boxed{1976}$

3. [**4**] Compute
$$\sum_{n=1}^{\infty} \frac{n+1}{n^2(n+2)^2}$$
Your answer in simplest form can be written as a/b, where a, b are relatively-prime positive integers. Find $a+b$.

Solution: Let K be the result.
$$\begin{aligned}
K &= \sum_{n=1}^{\infty} \frac{n+1}{n^2(n+2)^2} \\
4K + \sum_{n=1}^{\infty} \frac{1}{(n+2)^2} &= \sum_{n=1}^{\infty} \frac{n^2+4n+4}{n^2(n+2)^2} \\
4K - 1 - \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{(n+2)^2}{n^2(n+2)^2} \\
4K - \frac{5}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\
4K - \frac{5}{4} &= 0 \\
K &= \frac{5}{16}
\end{aligned}$$
So $a+b = 5+16 = 21$.

Answer: $\boxed{21}$

4. [**4**] Let f be a polynomial of degree 3 with integer coefficients such that $f(0) = 3$ and $f(1) = 11$. If f has exactly 2 integer roots, how many such polynomials f exist?

Solution: The answer is 0, but the argument is more general: if $f(0)$ and $f(1)$ are odd, then we claim that f can't have any integer roots:
Suppose a is an integer solution. Then $f(x) = (x-a)g(x)$, and $g(x)$ also has integer coefficients. So $f(0) = -ag(0)$ and $f(1) = (1-a)g(1)$, where $g(0)$ and $g(1)$ are also integers. Since either a or $1-a$ is even, $f(0)$ and $f(1)$ can't be both odd, as in the hypothesis, so f has no integer roots.

Answer: $\boxed{0}$

5. [**5**] What is the smallest natural number n greater than 2012 such that the polynomial $f(x) = (x^6+x^4)^n - x^{4n} - x^6$ is divisible by $g(x) = x^4+x^2+1$?

Solution: Let $g = x^4+x^2+1 = (x^2+x+1)(x^2-x+1)$. If a is solution to the polynomial $g_1 = x^2+x+1$ and b is solution to $g_2 = x^2-x+1$, then f is divisible by $g \iff a$ and b are solutions to f, as well, so $f(a) = f(b) = 0$.
On the other hand, if $a^2+a+1 = 0$, then $a^3 = 1$; also, since $b^2-b+1 = 0$, it follows that $b^3 = -1$.
$$\begin{aligned}
f(a) &= (a^6+a^4)^n - a^{4n} - a^6 = (1+a)^n - a^n - 1 \\
&= (-a^2)^n - a^n - 1 = 0
\end{aligned}$$

Noting that we can substitute every a^3 with 1,

$$f(a) = 0 \iff n \in \{6k+1, 6k+5\}$$

Similarly,

$$\begin{aligned} f(b) &= (b^6+b^4)^n - b^{4n} - b^6 = (1-b)^n + (-1)^n b^n - 1 \\ &= (-1)^n b^{2n} - (-1)^n b^n - 1 = 0 \end{aligned}$$

Noting that we can substitute every b^3 with -1,

$$f(b) = 0 \iff n \in \{6k+1, 6k+5\}$$

So the smallest instance of n larger than 2012 is 2015. This is the final answer.

Answer: $\boxed{2015}$
Thanks to Calvin Deng for pointing out the error in the original solutions with $f(b)$.

6. [**6**] Let a_n be a sequence such that $a_0 = 0$ and:

$$\begin{aligned} a_{3n+1} &= a_{3n} + 1 = a_n + 1 \\ a_{3n+2} &= a_{3n} + 2 = a_n + 2 \end{aligned}$$

for all natural numbers n. How many n less than 2012 have the property that $a_n = 7$?

Solution: Let $f(n) = a_n$.
In the end we obtain that $f(n)$ represents the sum of the digits of n in the representation in base 3. To reach that, first we see that:

$$n = 3\lfloor\frac{n}{3}\rfloor + r,$$

where $r \in \{1, 2\}$. Thus,

$$\begin{aligned} f(n) &= f(3\lfloor\frac{n}{3}\rfloor + r) = f(3\lfloor\frac{n}{3}\rfloor) + r \\ &= f(\lfloor\frac{n}{3}\rfloor) + r = f(\lfloor\frac{n}{3}\rfloor) + n - 3\lfloor\frac{n}{3}\rfloor \end{aligned}$$

We can also apply this to $\lfloor\frac{n}{3}\rfloor$ instead of n, taking into consideration that $\lfloor\frac{\lfloor\frac{n}{3}\rfloor}{3}\rfloor = \lfloor\frac{n}{3^2}\rfloor$. Thus,

$$\begin{aligned} f(n) &= f(\lfloor\frac{n}{3}\rfloor) + n - 3\lfloor\frac{n}{3}\rfloor \\ &= f(\lfloor\frac{n}{3^2}\rfloor) + \lfloor\frac{n}{3}\rfloor - 3\lfloor\frac{n}{3^2}\rfloor + n - 3\lfloor\frac{n}{3}\rfloor \\ &= f(\lfloor\frac{n}{3^2}\rfloor) + n - 2\lfloor\frac{n}{3}\rfloor - 3\lfloor\frac{n}{3^2}\rfloor \\ &= ... = f(\lfloor\frac{n}{3^k}\rfloor) + n - 2\lfloor\frac{n}{3}\rfloor - 2\lfloor\frac{n}{3^2}\rfloor - ... \\ &\qquad ... - 2\lfloor\frac{n}{3^{k-1}}\rfloor - 3\lfloor\frac{n}{3^k}\rfloor \end{aligned}$$

For k large enough, $\lfloor \frac{n}{2^k} \rfloor = 0$, so $f(n) = n - 2\lfloor \frac{n}{3} \rfloor - 2\lfloor \frac{n}{3^2} \rfloor - ...$, which is exactly the sum of the digits of n written in base 3.

So the problem now is just how many numbers less than 2012 have, in base 3, the sum of their digits 7, which is easy to find.

We will do the counting in base 3. Number 2012 in base 3 is 2202112, so every number smaller than 2012 will need to have at most 7 digits.
Case 1: how many numbers have 3 of its digits 2, one digit 1 and three 0?
We assume we have 7 positions and each digits occupies one of these positions (note: 0 can be the first digit, and we just neglect it). This arrangement can be done in $\binom{7}{3}\binom{4}{1}$ ways; from these, we have to subtract the numbers greater than 2202112. If one such number starts with 222, the rest can be continued in $\binom{4}{1}$ ways. If the number starts with 221, the rest can be completed in $\binom{4}{1}$ ways. Thus, there are 132 numbers smaller than 2202112 with three of the digits 2.
Case 2: how many numbers have 2 of their digits 2, 3 digits 1 and 2 digits 0?
Again, we assume we have to fill in the 7 positions. That can be done in $\binom{7}{2}\binom{5}{2}$ ways. From these numbers we have to subtract the ones larger than 2202112, thus the ones that start with 221. Those that start with 221 can be completed in $\binom{4}{2}$ ways, so we get that there are 204 numbers smaller than 2202112 which have two of their digits 2.
Case 3: how many numbers have one of their digits 2, five digits 1 and one 0?
We can form $\binom{7}{1}\binom{6}{1}$ numbers, and all of them are smaller than 2202122, so we get 42 numbers.
Case 4: how many numbers have no digit 2 and seven digits 1?
Only one, 1111111.
By adding, we get $132 + 204 + 42 + 1 = 379$ numbers which have the sum of their digits in base 3 seven, and are smaller than 2012. Answer: $\boxed{379}$

7. [**7**] Let a_n be a sequence such that $a_1 = 1$ and $a_{n+1} = \lfloor a_n + \sqrt{a_n} + \frac{1}{2} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. What are the last four digits of a_{2012}?

   Solution: Computing some particular cases suggests that the function f is defined by the following:

$$a_n = 1 + \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor,$$

   for all natural numbers n.
   We will show this hypothesis is true by induction. We assume it's true for n and prove it for $n + 1$.
   If $n = 2m$, then $a_n = 1 + m^2$. Thus,

$$\begin{aligned} a_{n+1} &= \lfloor a_n + \sqrt{a_n} + \frac{1}{2} \rfloor = \lfloor m^2 + 1 + \sqrt{m^2 + 1} + \frac{1}{2} \rfloor \\ &= m^2 + 1 + m = 1 + m(m + 1) = 1 + \lfloor \frac{n+1}{2} \rfloor \lfloor \frac{n+2}{2} \rfloor \end{aligned}$$

If $n = 2m+1$, then $a_n = 1 + m(m+1)$, thus

$$\begin{aligned} a_{n+1} &= \lfloor 1 + m(m+1) + \sqrt{1 + m(m+1)} + \frac{1}{2} \rfloor = 1 + m(m+1) + m + 1 \\ &= 1 + (m+1)^2 = 1 + \lfloor \frac{n+1}{2} \rfloor \lfloor \frac{n+2}{2} \rfloor \end{aligned}$$

The induction is complete, so that is the solution to the equation. In conclusion, $a_{2012} = 1 + \lfloor \frac{2012}{2} \rfloor \lfloor \frac{2013}{2} \rfloor = 1012037$, so the answer is 2037.

Answer: $\boxed{2037}$

8. [**8**] If n is an integer such that $n \geq 2^k$ and $n < 2^{k+1}$, where $k = 1000$, compute the following:

$$n - \left(\left\lfloor \frac{n - 2^0}{2^1} \right\rfloor + \left\lfloor \frac{n - 2^1}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{n - 2^{k-1}}{2^k} \right\rfloor \right).$$

Solution: Let us prove that $\lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n-2}{2^2} \rfloor + ... + \lfloor \frac{n-2^{k+1}}{2^k} \rfloor = n - k - 1$.
If we write $n = 2^k + a_{k-1}2^{k-1} + ... + a_1 2 + a_0$, then $\lfloor \frac{n-1}{2} \rfloor = 2^{k-1} + a_{k-1}2_{k-2} + ... + a_1 + a_0 - 1$, $\lfloor \frac{n-2}{2^2} \rfloor = 2^{k-2} + a_{k-1}2^{k-3} + ... + a_1 - 1$, ..., $\lfloor \frac{n-2^{k-1}}{2^k} \rfloor = 1 + a_{k-1} - 1$.
By adding all the terms, we obtain:

$$\begin{aligned} \sum_{i=1}^{k} \lfloor \frac{n - 2^{i-1}}{2^i} \rfloor &= (1 + 2 + ... + 2^{k-1}) + a_{k-1}(1 + 2 + ... + 2^{k-2} + ... \\ &\quad ... + a_1 + (a_{k-1} - 1) + (a_{k-2} - 1) + ... + (a_0 - 1) \\ &= 2^k - 1 + a_{k-1}(2^{k-1} - 1) + ... + a_1 + (a_{k-1} + a_{k-2} + ... + a_0) - k - 1 \\ &= 2^k + a_{k-1}2^{k-1} + ... + a_1 2 + a_0 - k - 1 = n - k - 1 \end{aligned}$$

Thus the answer is $k + 1$ or 1001.

Answer: $\boxed{1001}$