PUMaC 2012

Algebra B Solutions

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1. [**3**] Find the largest n such that the last nonzero digit of $n!$ is 1.

   Solution: For $n > 2$ if we express $n! = 2^i 5^j m$ where, $i, j, m \in \mathbb{N}$ such that m is not divisible by 2 or 5, we can easily observe that $i > j$. Thus, we can write $n! = 2^{i-j} 10^j m$ which means that the last non-zero digit of $n!$ is always going to be an even number. Thus, the only possible values for n are 0 and 1, which both give the last non-zero digit equal to 1. Because 1 is greater, the final answer is 1.

   Answer: $\boxed{1}$

2. [**3**] Define a sequence a_n such that $a_n = a_{n-1} - a_{n-2}$. Let $a_1 = 6$ and $a_2 = 5$. Find $\sum_{n=1}^{1000} a_n$.

   Solution: $a_1 = 6$, $a_2 = 5$, $a_3 = -1$, $a_4 = -6$, $a_5 = -5$, $a_6 = 1$, $a_7 = 6$, $a_8 = 5$, so the sequence starts to repeat itself. The sum of the first 6 terms is 0, so the sum of the first 996 terms is 0; thus, the sum of the first 1000 terms is $6 + 5 + (-1) + (-6) = 4$.

   Answer: $\boxed{4}$

3. [**4**] Evaluate $\sqrt[3]{26 + 15\sqrt{3}} + \sqrt[3]{26 - 15\sqrt{3}}$.

   Solution: Let $x = \sqrt[3]{26 + 15\sqrt{3}}$, $y = \sqrt[3]{26 - 15\sqrt{3}}$. We are looking for $z = x + y$. We can do this by cubing $x + y$.

$$
\begin{aligned}
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x+y)^3 &= 26 + 15\sqrt{3} + 26 - 15\sqrt{3} + 3\sqrt[3]{26^2 - 3 * 15^2}(x+y) \\
(x+y)^3 &= 52 + 3\sqrt[3]{1}(x+y) \\
z^3 &= 52 + 3z \\
z^3 - 3z - 52 &= 0 \\
(z-4)(z^2 + 4z + 13) &= 0
\end{aligned}
$$

   Since $z^2 + 4z + 13$ has no real root, z has to equal 4.

   Answer: $\boxed{4}$

4. [**4**] If x, y, and z are real numbers with $\frac{x-y}{z} + \frac{y-z}{x} + \frac{z-x}{y} = 36$, find

$$2012 + \frac{x-y}{z} \cdot \frac{y-z}{x} \cdot \frac{z-x}{y}$$

Solution:

$$\begin{aligned}36 &= \frac{(x-y)xy+(y-z)yz+(z-x)xz}{xyz}\\ &= \frac{(x-y)xy+y^2z-yz^2+xz^2-x^2z}{xyz}\\ &= \frac{(x-y)xy-(x+y)(x-y)z+z^2(x-y)}{xyz}\\ &= \frac{(x-y)(xy-xz-yz+z^2)}{xyz}\\ &= \frac{(x-y)(y-z)(x-z)}{xyz}\\ 36 &= -\frac{(x-y)(y-z)(z-x)}{xyz}\end{aligned}$$

$$2012+\frac{(x-y)(y-z)(z-x)}{xyz} = 2012-36 = 1976$$

Answer: $\boxed{1976}$

5. [**5**] Considering all numbers of the form $n = \lfloor \frac{k^3}{2012} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and k ranges from 1 to 2012, how many of these n's are distinct?

   Solution:

$$\frac{(k+1)^3}{2012}-\frac{k^3}{2012} = \frac{3k^3+3k+1}{2012}$$
$$\frac{3k^3+3k+1}{2012} \geq 1 \iff 3k^2+3k+1 \geq 2012$$
$$\iff 3k^2+3k \geq 2011 \iff 3k^2+3k \geq 2010$$
$$\iff k^2+k \geq 670 \iff k \geq 26$$

   Thus, for k of at least 26, the difference between two consecutive fractions is at least 1, so the difference between their integer parts is also at least 1, so the numbers are different; in conclusion, for k between 26 and 2012, there are 1987 different numbers.
   For k less than 26, the difference between two consecutive fractions is less than 1, so the integer parts of two consecutive fractions is at most 1. For $k = 1$, $[\frac{k^3}{2012}] = 0$, and for $k = 25$, $[\frac{k^3}{2012}] = 7$, so for k less than 26, there are 8 different numbers.
   In the end, we get that for k ranging from 1 to 2012, there are 1995 different numbers in the sequence.

   Answer: $\boxed{1995}$

6. [**6**] Compute

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2(n+2)^2}$$

Your answer in simplest form can be written as a/b, where a, b are relatively-prime positive integers. Find $a + b$.

Solution: Let K be the result.

$$\begin{aligned} K &= \sum_{n=1}^{\infty} \frac{n+1}{n^2(n+2)^2} \\ 4K + \sum_{n=1}^{\infty} \frac{1}{(n+2)^2} &= \sum_{n=1}^{\infty} \frac{n^2+4n+4}{n^2(n+2)^2} \\ 4K - 1 - \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{(n+2)^2}{n^2(n+2)^2} \\ 4K - \frac{5}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\ 4K - \frac{5}{4} &= 0 \\ K &= \frac{5}{16} \end{aligned}$$

So $a + b = 5 + 16 = 21$.

Answer: $\boxed{21}$

7. [**7**] Let f be a polynomial of degree 3 with integer coefficients such that $f(0) = 3$ and $f(1) = 11$. If f has exactly 2 integer roots, how many such polynomials f exist?

Solution: The answer is 0, but the argument is more general: if $f(0)$ and $f(1)$ are odd, then we claim that f can't have any integer roots:
Suppose a is an integer solution. Then $f(x) = (x-a)g(x)$, and $g(x)$ also has integer coefficients. So $f(0) = -ag(0)$ and $f(1) = (1-a)g(1)$, where $g(0)$ and $g(1)$ are also integers. Since either a or $1-a$ is even, $f(0)$ and $f(1)$ can't be both odd, as in the hypothesis, so f has no integer roots.

Answer: $\boxed{0}$

8. [**8**] Let a_n be a sequence such that $a_1 = 1$ and $a_{n+1} = \lfloor a_n + \sqrt{a_n} + \frac{1}{2} \rfloor$. What are the last four digits of a_{2012}?

Solution: Computing some particular cases suggests that the function f is defined by the following:

$$a_n = 1 + \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor,$$

for all natural numbers n.
We will show this hypothesis is true by induction. We assume it's true for n and prove it for $n+1$.

If $n = 2m$, then $a_n = 1 + m^2$. Thus,

$$a_{n+1} = \lfloor a_n + \sqrt{a_n} + \frac{1}{2} \rfloor = \lfloor m^2 + 1 + \sqrt{m^2+1} + \frac{1}{2} \rfloor$$
$$= m^2 + 1 + m = 1 + m(m+1) = 1 + \lfloor \frac{n+1}{2} \rfloor \lfloor \frac{n+2}{2} \rfloor$$

If $n = 2m + 1$, then $a_n = 1 + m(m+1)$, thus

$$a_{n+1} = \lfloor 1 + m(m+1) + \sqrt{1 + m(m+1)} + \frac{1}{2} \rfloor = 1 + m(m+1) + m + 1$$
$$= 1 + (m+1)^2 = 1 + \lfloor \frac{n+1}{2} \rfloor \lfloor \frac{n+2}{2} \rfloor$$

The induction is complete, so that is the solution to the equation. In conclusion, $a_{2012} = 1 + \lfloor \frac{2012}{2} \rfloor \lfloor \frac{2013}{2} \rfloor = 1012037$, so the answer is 2037.

Answer: $\boxed{2037}$