Combinatorics A Solutions

Written by Andy Zhu

1. There are 8 even numbers and 7 odd numbers from 16 to 30. For the sum of three integers to be even, either all three must be even, or two must be odd and the last must be even. There are $\binom{8}{3} = 7 \cdot 8$ ways to choose the three even numbers, and $8 \cdot \binom{7}{2} = 7 \cdot 24$ ways to choose the one even and two odd integers. In total, there are $\binom{15}{3} = 7 \cdot 65$ ways to choose three distinct numbers from 15 numbers. Thus, the probability that the sum is even is $\frac{7\cdot 8+7\cdot 24}{7\cdot 65} = \frac{32}{65}$, and the answer is $m + n = \boxed{97}$.

2. We can do this by casework. Notice that there is exactly one fixed element between consecutive permutations, and there are exactly two permutations for each fixed digit as a fixed element. Without loss of generality, suppose the first permutation is given by (123) (six possibilities), and that the second permutation fixes the third element, and so must be (213) (three possibilities). Without loss of generality again, one may assume that the third permutation fixes the second element (between the first and second elements), and so must be (312) (two possibilities). For the fourth permutation, there are two distinct cases: either the first or third elements are fixed.

   - In the former case, one gets the permutation (213). The two permutations left are (231) and (321) and must occur in that order.
   - In the latter case, one gets the permutation (321). The two permutations left are (231) and (213) and must occur in that order.

   Hence, there are $6 \times 3 \times 2 \times (1 + 1) = \boxed{72}$ possible orderings of these permutations.

   An equivalent restatement of the problem is to count the number of Hamiltonian paths of the complete bipartite graph $K_{3,3}$ (each bipartition corresponding to a copy of $A_3 \leq S_3$), which is readily seen to be $6 \times 3 \times 2 \times 2 = \boxed{72}$.

   Problem contributed by Luke Paulsen.

3. Note that there is some symmetry: the probability of rolling a 9 is the same for both two rolls of the first die and two rolls of the second dice (consider the correspondence of a face f of the first die with that of a face $9 - f$ on the second die), and is given by $4/36 = 1/9$ by listing the possible face combinations. If the first and second dice are rolled, the probability of rolling 9 is $6/36 = 1/6$. By conditional probability, the answer is

$$P(\text{same dice} \mid \text{rolled } 9) = \frac{P(\text{rolled } 9 \mid \text{same dice})}{P(\text{rolled } 9)} = \frac{1/9}{\frac{1}{2}\cdot(1/9) + \frac{1}{2}\cdot(1/6)} = \frac{2}{5},$$

   so the answer is $2 + 5 = \boxed{7}$.

4. Consider the general case for $P_i = (x_i, y_i) \in \mathbb{Z}_N^2$, where here $N = 7$. For each value of M with $0 \leq M \leq N$, there are $\binom{N}{M}$ ways to pick each of the x- and y-coordinates of the M points,

since the M x- and y-coordinates are distinct integers between 0 and N, inclusive. Hence, the answer is

$$\sum_{M=0}^{N} \binom{N}{M}^2 = \binom{2N}{N} = \binom{14}{7} = \boxed{3432}$$

by Vandermonde's identity. Alternatively, consider the following bijection to the number of paths from $(0, -1)$ to $(N, N-1)$ travelling only up or right at lattice points: keep travelling right until you reach one of the values of x_i, at which point you move up to P_i and repeat. Conversely, each such path corresponds to the set of points where an 'up' step transitions to a 'right' step. There are $\frac{(2N)!}{N!N!}$ such paths.

5. Call the last person at the beginning A. If A is at ith position, for the next round, there is $\frac{i-1}{5}$ chance that the first person remains in front of A, which means the position of A remains the same. Otherwise, A's position advances by 1. Hence the expected number of rounds for A to move the position by 1 is

$$\begin{aligned} \sum_{k=1}^{+\infty} \left(\frac{i-1}{5}\right)^{k-1} \left(1 - \frac{i-1}{5}\right) k &= \sum_{k=0}^{+\infty} \left(\frac{i-1}{5}\right)^{k} (k+1) - \sum_{k=0}^{+\infty} \left(\frac{i-1}{5}\right)^{k} k \\ &= \sum_{k=0}^{+\infty} \left(\frac{i-1}{5}\right)^{k} = \frac{1}{1 - \frac{i-1}{5}} = \frac{5}{5-i+1} \end{aligned}$$

(Note that this is the expectation of a geometric distribution with parameter $p = \frac{5-i+1}{5}$.) So in order to move A to the front, the total expected number of rounds needed is

$$\sum_{i=2}^{5} \frac{5}{5-i+1} = \frac{5}{1} + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} = \frac{125}{12},$$

so the answer is $125 + 12 = \boxed{137}$.

Problem contributed by Xufan Zhang.

6. We do casework on the number of faces colored black. The number of rotationally distinguishable colorings when k faces are colored black is the same as that when $10 - k$ faces are black, so it suffices to compute when $k = 0, 1, 2, 3, 4, 5$. Although tedious, most of the cases turn out to be very similar. Indeed, the dual of the shape is a pentagonal prism, so equivalently, the question asks for the number of 2-vertex-colorings of a pentagonal prism (which is how we'll describe the shape in the following). Without loss of generality, we can fix the orientation of the prism such that the first black vertex to consider lies on the "top" face.

- $k = 0$: 1 way.
- $k = 1$: 1 way.
- $k = 2$: If both black vertices lie on the same pentagonal face, there are 2 ways. Otherwise, the first vertex fixes the "top" face and the second vertex can be any of the 5 vertices of the bottom face, for 7 ways.

- $k = 3$: 3 vertices on a face can happen in 2 ways. 2 vertices on one face can occur in 2 ways, and each way fixes the top face, so the third vertex can be any of the 5 vertices of the bottom face, for 12 ways.
- $k = 4$: If all four vertices lie on a face, there is 1 way. If three lie on a face, there are 2 ways, and the fourth vertex can be any of the 5 vertices of the other face. If two lie on one face and two on the other, we need to be a bit careful: there are 5 if both pairs are adjacent, 5 if neither pair is adjacent, and 5 if one pair is adjacent and the other is non-adjacent. This gives a total of $1 + 2 \times 5 + 5 + 5 + 5 = 26$ ways.
- $k = 5$: If all five vertices lie on a face, there is 1 way. If four lie on a face, there is 1 way, and the fourth vertex can be any of the 5 vertices of the other face. If three lie on a face and two on the other, there are 2 ways to arrange the three vertices on the top face and $\binom{5}{2} = 10$ ways to position the bottom two, for a total of $1 + 5 + 2 \times 10 = 26$ ways.

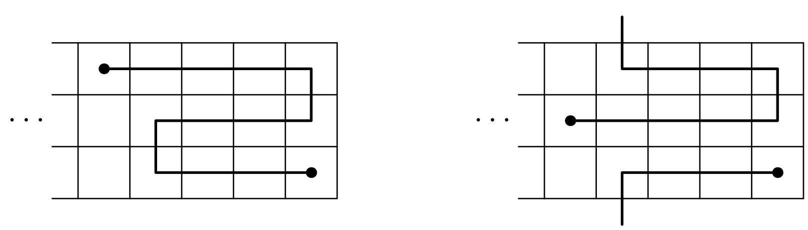
Summing, there are $2(1 + 1 + 7 + 12 + 26) + 26 = \boxed{120}$ arrangements.

Alternatively, we can count the symmetries of this 3D shape. There is 1 identity rotation, 4 $(72°, 144°, 216°, 288°)$ rotations about the apex vertices, and 5 180° rotations about the non-apex vertices that send the shape to itself, and these form the symmetry group of the figure. Respectively, these fix $10, 2$, and 5 orbits of faces. By the (not-)Burnside's lemma, the answer is $\frac{2^{10}+4\cdot 2^2+5\cdot 2^5}{10} = \boxed{120}$.

7. Equivalently, consider a Hamiltonian walk on the squares of a cylindrical 3×40 chessboard (wrapped so that the two long sides overlap). Let A_n denote the number of ways to start at the top left corner $(1, 3)$ of such a $3 \times n$ cylindrically-wrapped chessboard and end in the bottom right corner $(n, 1)$, and let B_n denote the number of ways to start at the top left corner $(1, 3)$ and end in the top right corner $(n, 3)$. Then the question is asking for the value of B_{40}. The key is the two recurrence relations

$$A_n = 1 + \sum_{i=0}^{n-1} A_i + B_i, \quad B_n = 2\sum_{i=0}^{n-1} A_i \qquad (*)$$

This comes from the observation that for any path starting on the left side and ending on the right side, if there were k horizontal steps before reaching the destination point on the right side, then the previous $2k + 1$ moves before those are also fixed upto two possibilities:



This holds true for each value of k from 0 to $n - 2$. Note that that the number of paths from column 1 to column n is then reduced to counting the number of paths from column 1 to column $n - k - 1$, upto a change in the row. The one exception is when $k = n - 1$, in which case we end up with a valid path of the entire $3 \times n$ board if the destination is not in the first row (hence the $+1$ in $(*)$). When we keep track of the row change, we obtain the recursions in $(*)$ - recall the cylindrical nature of the chessboard means that there is symmetry in the cases of the bottom two rows.

It thus suffices to solve $(*)$. Notice that

$$A_{n+1} - A_n = A_n + B_n, \quad B_{n+1} - B_n = 2A_n.$$

Substituting the first equation into the second one gives

$$(A_{n+2} - 2A_{n+1}) - (A_{n+1} - 2A_n) = 2A_n \implies A_{n+2} = 3A_{n+1}.$$

Hence, for all $n > 1$, one has the explicit formula $A_n = 3^{n-2}A_2$. With a bit of casework and visualization, we indeed see that $A_1 = 1, A_2 = 2, A_3 = 6, \cdots$, so $A_n = 2 \cdot 3^{n-2}$ for all $n > 1$. Thus, the desired answer is the value of $A_{40} = 2 \cdot 3^{38} \pmod{100}$. Since $\varphi(100) = \frac{1}{2} \cdot \frac{4}{5} \cdot 100 = 40$, Euler's totient theorem shows that $3^{40} \equiv 1 \pmod{100}$. Also, we note that $1 \equiv 201 \equiv 3 \cdot 67 \pmod{100}$, so

$$A_{40} = 2 \cdot 3^{38} \equiv 2 \cdot \left(3^{-1}\right)^2 \equiv 2 \cdot 67^2 \equiv 8978 \equiv \boxed{78} \pmod{100}.$$

8. Consider the permutation $\alpha = (1854)(2367)$ corresponding to the mapping $x \mapsto 3^{-1}(x-1) \equiv 3x - 3$ on $\mathbb{Z}/8\mathbb{Z}$. The problem asks for the number of possible permutations $\pi = \beta^{-1}\alpha\beta$ as β ranges over S_8. The operation $\pi(\beta) = \beta^{-1}\alpha\beta$ is called the conjugation of α with respect to β. The key fact here is that two permutations are conjugate if and only if they have the same cycle structure. Thus, π also has two 4-cycles. There are $\frac{1}{2}\binom{8}{4} = 35$ ways to pick the elements of one of the 4-cycles (the 1/2 since the order of the two 4-cycles does not matter), and $\frac{4!}{4} = 6$ ways to permute each of the two 4-cycles upto cyclic rotation. The answer is $35 \times 6^2 = \boxed{1260}$.

   We sketch a proof of the fact that conjugation preserves cycle structure. View each permutation as a function on $\{1, 2, \ldots, 8\}$. Consider first when $\alpha = (a_1\ a_2\ \ldots\ a_n)$ is a cycle. We claim that $\alpha\beta = \beta\pi$ where π is the cycle $(\beta(a_1)\ \beta(a_2)\ \ldots\ \beta(a_n))$. Indeed, $\beta(\alpha(a_i)) = \beta(a_{i+1}) = \pi(\beta(a_i))$, so $\alpha\beta$ and $\beta\pi$ agree on all of the a_i's. For any other symbol b not in the a_i's, b is fixed by α and $\beta(b)$ is fixed by π, so $\beta(\alpha(b)) = \beta(b) = \pi(\beta(b)$, so $\alpha\beta$ and $\beta\pi$ agree on all the symbols outside of the a_i's. To finish, we note that any cycle α can be written as a composition of disjoint cycle permutations.