



Combinatorics B

1. [3] Your friend sitting to your left (or right?) is unable to solve any of the eight problems on his or her Combinatorics B test, and decides to guess random answers to each of them. To your astonishment, your friend manages to get two of the answers correct. Assuming your friend has equal probability of guessing each of the questions correctly, what is the average possible value of your friend's score? Recall that each question is worth the point value shown at the beginning of each question.
2. [3] If the probability that the sum of three distinct integers between 16 and 30 (inclusive) is even can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.
3. [4] How many ways are there to arrange the 6 permutations of the tuple $(1, 2, 3)$ in a sequence, such that each pair of adjacent permutations contains at least one entry in common? For example, a valid such sequence is given by

$$(3, 2, \mathbf{1}) - (\mathbf{2}, 3, 1) - (2, 1, \mathbf{3}) - (\mathbf{1}, 2, 3) - (1, 3, \mathbf{2}) - (3, 1, 2).$$

4. [4] For a set S of integers, define $\max(S)$ to be the maximal element of S . How many non-empty subsets $S \subseteq \{1, 2, 3, \dots, 10\}$ satisfy $\max(S) \leq |S| + 2$?
5. [5] Jim has two fair 6-sided dice, one whose faces are labelled from 1 to 6, and the second whose faces are labelled from 3 to 8. Twice, he randomly picks one of the dice (each die equally likely) and rolls it. Given the sum of the resulting two rolls is 9, if $\frac{m}{n}$ is the probability he rolled the same die twice where m, n are relatively prime positive integers, then what is $m + n$?
6. [6] How many (possibly empty) sets of lattice points $\{P_1, P_2, \dots, P_M\}$, where each point $P_i = (x_i, y_i)$ for $x_i, y_i \in \{0, 1, 2, 3, 4, 5, 6\}$, satisfy that the slope of the line $P_i P_j$ is positive for each $1 \leq i < j \leq M$? An infinite slope, e.g. P_i is vertically above P_j , does not count as positive.
7. [7] 5 people stand in a line facing one direction. In every round, the person at the front moves randomly to any position in the line, including the front or the end. Suppose that $\frac{m}{n}$ is the expected number of rounds needed for the last person of the initial line to appear at the front of the line, where m and n are relatively prime positive integers. What is $m + n$?
8. [8] A PUMaC grader is grading the submissions of forty students s_1, s_2, \dots, s_{40} for the individual finals round, which has three problems. After grading a problem of student s_i , the grader either:
 - grades another problem of the same student, or
 - grades the same problem of the student s_{i-1} or s_{i+1} (if $i > 1$ and $i < 40$, respectively).

He grades each problem exactly once, starting with the first problem of s_1 and ending with the third problem of s_{40} . Let N be the number of different orders the grader may grade the students' problems in this way. Find the remainder when N is divided by 100.