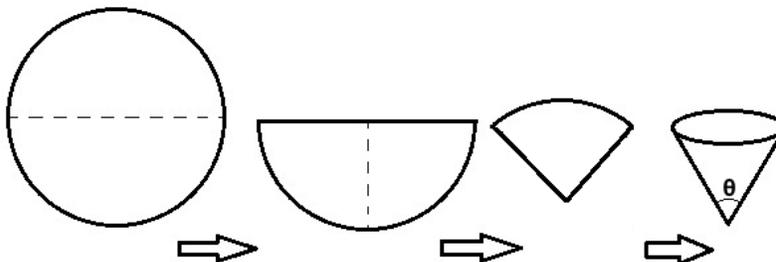




## Geometry B Solutions

Written by Ante Qu

1. [3] During chemistry labs, we oftentimes fold a disk-shaped filter paper twice, and then open up a flap of the quartercircle to form a cone shape, as in the diagram. What is the angle  $\theta$ , in degrees, of the bottom of the cone when we look at it from the side?



Solution:

Let  $r$  be the radius of the original circle of the paper. Since we are opening up the second fold, the opening is made from the semicircle from the first fold. Thus the perimeter of the opening is  $\pi r$ . Therefore the radius of its opening is  $r/2$ . Draw a right triangle with one side as the altitude from the vertex to the opening, another side as the radius of the opening, and the third side as a slant, which is the radius of the original paper. Considering the angle between the slant and the altitude, we have that the hypotenuse is double the opposite. Thus this is a 30-60-90 triangle, with the angle between the slant and the altitude equal to 30 degrees. Hence when we look at the cone from the side, half of the bottom angle is 30 degrees, and so the total angle is  $\boxed{60}$  degrees.

Problem contributed by Xufan Zhang

2. [3] A 6-inch-wide rectangle is rotated 90 degrees about one of its corners, sweeping out an area of  $45\pi$  square inches, excluding the area enclosed by the rectangle in its starting position. Find the rectangle's length in inches.

Solution: After setting up the problem, we can see that the swept-out area is equal to a quartercircle with the rectangle's diagonal as its radius. Since  $45\pi$  square inches is the area of the quartercircle, the circle's radius is  $6\sqrt{5}$  inches. Using the Pythagorean Theorem, we have that the length equals  $\sqrt{180 - 36} = \boxed{12}$  inches.

Problem contributed by Luke Paulsen

3. [4] Let  $A$  be a regular 12-sided polygon. A new 12-gon  $B$  is constructed by connecting the midpoints of the sides of  $A$ . The ratio of the area of  $B$  to the area of  $A$  can be written in simplest form as  $(a + \sqrt{b})/c$ , where  $a, b, c$  are integers. Find  $a + b + c$ .

Solution: Note that the apothem of  $A$  is the circumradius of  $B$ , and  $A$  and  $B$  are similar. Therefore the ratio of the area of  $B$  to the area of  $A$  is the square of the ratio of the circumradius of  $B$  to the circumradius of  $A$ . Draw a right triangle with an apothem, half a side, and a



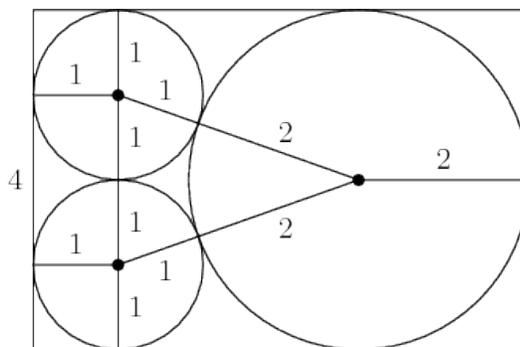
circumradius of  $A$ . A 12-gon has an interior angle of  $(\frac{10}{12})180^\circ = 150^\circ$ , so the angle between the side and the circumradius is  $75^\circ$ . Therefore the ratio of the apothem (circumradius of  $B$ ) to the circumradius (of  $A$ ) is  $\sin 75^\circ$ . Using the angle addition formula,

$$\begin{aligned}\sin 75^\circ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \frac{1}{2} \\ \sin 75^\circ &= \frac{\sqrt{2}(1 + \sqrt{3})}{4}\end{aligned}$$

To find the ratio of the areas, we square this ratio, which becomes  $(2 + \sqrt{3})/4$ . So  $a + b + c = \boxed{9}$ .

4. [4] Three circles,  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$ , are externally tangent to each other, with radii of 1, 1, and 2 respectively. Quadrilateral  $ABCD$  contains and is tangent to all three circles. Find the minimum possible area of  $ABCD$ . Your answer will be of the form  $a + b\sqrt{c}$  where  $c$  is not divisible by any perfect square. Find  $a + b + c$ .

Solution: In order to have the smallest quadrilateral, we want to make it tangent at as many points as possible. Consider the following quadrilateral.



Note that 3 of the sides are fixed as 3 sides of a rectangle, so the area is proportional to the distance from the side tangential to the two small circles to the midpoint of the fourth side. This can be minimized by making the fourth side the fourth side of a rectangle, which is tangent to the large circle at its midpoint. To find the width of this rectangle, note that it is the sum of a radius of a small circle, a radius of the large circle, and the altitude of an isosceles triangle with side lengths of 2, 3, 3. Using the Pythagorean Theorem, we find the altitude to be  $2\sqrt{2}$ . To calculate the whole area of the rectangle, we multiply its height, 4, by its width,  $3 + 2\sqrt{2}$ , to get  $12 + 8\sqrt{2}$ . So  $a + b + c = 12 + 8 + 2 = \boxed{22}$ .

5. [5] Two circles centered at  $O$  and  $P$  have radii of length 5 and 6 respectively. Circle  $O$  passes through point  $P$ . Let the intersection points of circles  $O$  and  $P$  be  $M$  and  $N$ . The area of triangle  $\triangle MNP$  can be written in the form  $a/b$ , where  $a$  and  $b$  are relatively prime positive integers. Find  $a + b$ .

Solution: Draw  $\triangle MOP$  and  $\triangle NOP$ . They are each triangles with side lengths of 5, 5, 6. Let  $V$  be the point of intersection of  $\overline{MN}$  and  $\overline{OP}$ . Note that  $\overline{MV}$  and  $\overline{NV}$ , the altitudes from  $OP$



to vertices  $M$  and  $N$ , are each equal to twice the area of each triangle divided by the length of side  $\overline{OP}$ . To find the area of  $\triangle MOP$ , simply take the altitude to  $\overline{MP}$  and call the intersection at the base  $X$ . Since this triangle is isosceles, the altitude splits  $\overline{MP}$  into two equal segments, so  $\overline{XP}$  has a length of 3., so  $\triangle OXP$  forms a 3-4-5 right triangle, and the altitude has a length of 4. As a result,  $\overline{MV}$  has a length of  $24/5$ , and so  $MV = 48/5$ . Using the Pythagorean Theorem, we have that the length of  $\overline{VP}$  is  $18/5$ , and the total area of  $\triangle NOP$  is  $432/25$ , so  $a + b = 432 + 25 = \boxed{457}$ .

6. [6] A square is inscribed in an ellipse such that two sides of the square respectively pass through the two foci of the ellipse. The square has a side length of 4. The square of the length of the minor axis of the ellipse can be written in the form  $a + b\sqrt{c}$  where  $a$ ,  $b$ , and  $c$  are integers, and  $c$  is not divisible by the square of any prime. Find the sum  $a + b + c$ .

Solution: Let  $a$  be the length of the major axis,  $b$  be the length of the minor axis, and  $c$  be the distance from the foci to the center of the ellipse.

Since the sum of the distances from any point on the ellipse to the foci is  $2a$ , we can use a vertex of the square to calculate  $2a$ . We have

$$2a = 2 + 2\sqrt{5}$$

so  $a = 1 + \sqrt{5}$ . Now using the relation  $b^2 = a^2 - c^2$ , we have

$$b^2 = 1 + 2\sqrt{5} + 5 - 4 = 2 + 2\sqrt{5}$$

$$2b = 2\sqrt{2 + 2\sqrt{5}} = \sqrt{8 + 8\sqrt{5}}$$

$$a + b + c = 8 + 8 + 5 = \boxed{21}.$$

Problem contributed by Elizabeth Yang

7. [7] Assume the earth is a perfect sphere with a circumference of 60 units. A great circle is a circle on a sphere whose center is also the center of the sphere. There are three train tracks on three great circles of the earth. One is along the equator and the other two pass through the poles, intersecting at a 90 degree angle. If each track has a train of length  $L$  traveling at the same speed, what is the maximum value of  $L$  such that the trains can travel without crashing?

Solution: Let  $E$  be the circumference of the earth. Note that if no train barely-crashes into another train ("barely-crashing" refers to the case where a train enters an intersection just as the tail of another train barely leaves the intersection), then we can simply extend every train until one train barely-crashes into another. Therefore we can assume without loss of generality that train  $A$  is barely leaving the north pole just as train  $B$  is entering the north pole.

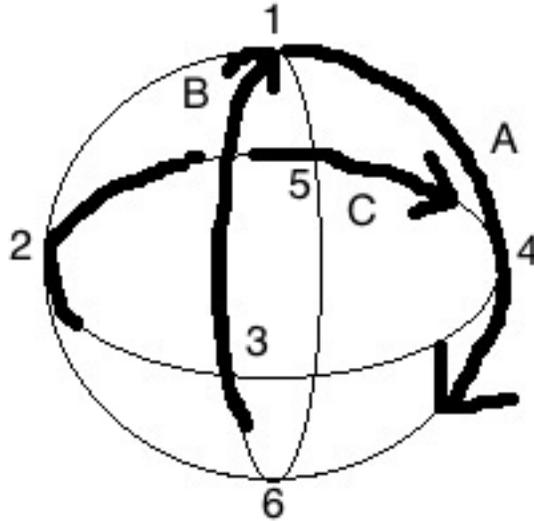
For the sake of clarity, we will label the intersections 1 through 6, with 1 and 6 being the north and south poles respectively, 4 and 2 on train  $A$ 's track, and 3 and 5 on train  $B$ 's track, as in the diagram on the next page.

Consider the possibility that the trains are longer than  $E/4$ . If we can make this work, then we no longer need to consider scenarios for trains to be shorter. Then at the moment the tail of train  $A$  barely leaves the north pole (and train  $B$  barely enters the north pole), train



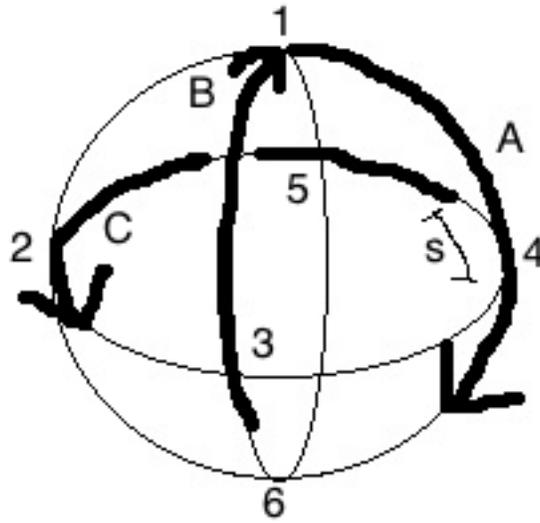
$C$  is either heading towards intersection 3 (where train  $B$  is) or heading towards intersection 4 (where train  $A$  is), on the longer  $3E/4$  portion of its track that passes through intersections 2 and 5.

We will first consider the case for when train  $C$  is heading towards intersection 4 (where train  $A$  sits):



In order for it to not collide into train  $A$ , it must be at least a distance of  $E/4$  away from intersection 4. In order to not collide into train  $B$  when train  $B$  comes down to intersection 5, train  $C$  must also either be of length at most  $E/4$  (if it initially sits with its front at intersection 5), in which case it will have left the intersection by the time train  $B$  arrives, or it must be much more than  $E/2$  away from intersection 4, so that it doesn't arrive at intersection 5 when train  $B$  arrives, which leaves much less than  $E/4$  room for it to exist (between intersections 2 and 3). Since we are considering the possibility when trains are longer than  $E/4$ , neither scenario is acceptable, as both cases give it at most  $E/4$  length. Therefore if the trains are longer than  $E/4$ , train  $C$  cannot be heading towards train  $A$  at intersection 4.

Now let's consider the other case where train  $C$  is heading towards intersection 3 (where train  $B$  sits) at the moment the tail of train  $A$  barely leaves the north pole. Let  $s$  denote the offset of train  $C$ 's tail from intersection 4:



In order for train  $C$  to not collide into train  $B$  at intersection 3, the distance between the front of train  $C$  and the intersection must be at least the distance between the tail of train  $B$  and the intersection:

$$3E/4 - (L + s) \geq L - E/4$$

In order for train  $B$  to not collide into train  $C$ , we simply have to make sure  $s$  is nonnegative (as in, the distance between the front of train  $B$  and intersection 5 (which is  $E/4$ ) must be at least the distance between the tail of train  $C$  and intersection 5 (which is  $E/4 - s$ ), so  $s$  simply has to be at least 0).

In order for train  $A$  to not collide with train  $C$  at intersection 2, the distance between the front of train  $A$  and the intersection must be at least the distance between the tail of train  $C$  and the intersection:

$$3E/4 - L \geq E/2 - s$$

The first inequality can be rearranged to become

$$L \leq \frac{E - s}{2}$$

and the second inequality can be rearranged to become

$$L \leq \frac{E + 4s}{4}$$

As  $s$  increases from 0 to  $E/2$ , the first bound decreases while the second increases, so the maximum occurs when the two are equal:

$$2E - 2s = E + 4s$$

$$s = \frac{E}{6}$$

$$L \leq \frac{5E}{12}$$

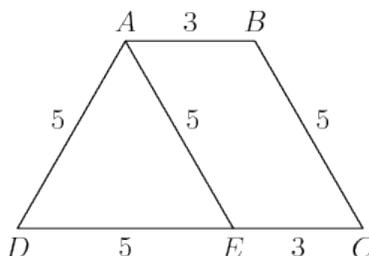


Since  $L = \frac{5E}{12}$  is the maximum length that satisfies both inequalities, this length is the maximum possible length when the length is greater than  $E/4$ . Note that when  $L = \frac{5E}{12}$ , if we advance time such that the trains move by  $s = E/6$ , we have the same exact situation as before, except with the new north pole at 3, train  $C$  becoming the new train  $B$ , train  $B$  becoming the new train  $A$ , and train  $A$  becoming the new train  $C$ , with the same offset  $s = E/6$ , and no collisions having occurred. Therefore this length of  $5E/12$  is possible with no collisions, and we no longer need to consider any cases where the train has a length of less than  $E/4$ . Plugging in  $E = 60$ , we have  $L = \boxed{25}$ .

Problem contributed by John Stogin

8. [8] A cyclic quadrilateral  $ABCD$  has side lengths  $AB = 3$ ,  $BC = AD = 5$ , and  $CD = 8$ . What's the radius of its circumcircle? Your answer can be written in the form  $a\sqrt{b}/c$ , where  $a, b, c$  are positive integers,  $a, c$  are relatively prime, and  $b$  is not divisible by the square of any prime. Find  $a + b + c$ .

Solution: Note that  $ABCD$  is an isosceles trapezoid with  $\overline{AB}$  parallel to  $\overline{CD}$ . Draw  $\overline{AE}$  parallel to  $\overline{BC}$  with  $E$  on segment  $\overline{CD}$ .



Since  $ABCE$  is a parallelogram,  $CE = 3$  and  $AE = 5$ , so  $DE = 5$ . Since  $AD = AE = DE = 5$ ,  $\triangle ADE$  is an equilateral triangle and so  $m\angle D = m\angle C = 60^\circ$ , and thus  $m\angle A = m\angle B = 120^\circ$ . Applying the law of cosines on  $\triangle ABD$  gets  $BD = \sqrt{9 + 25 + 15} = 7$ . The area of triangle  $\triangle ABD$  is equal to both  $\frac{1}{2}ab \sin 120^\circ$  and  $\frac{abc}{4R}$ , so  $2R \sin 120^\circ = c$  where  $c = 7$ . Solving for  $R$ , we have  $R = \frac{7\sqrt{3}}{3}$ , so  $a + b + c = \boxed{16}$ .

Problem contributed by Chengyue Sun

Thanks to teams for correcting the area formula.