PUMaC 2012

Number Theory A

1. [**3**] Albert has a very large bag of candies and he wants to share all of it with his friends. At first, he splits the candies evenly amongst his 20 friends and himself and he finds that there are five left over. Ante arrives, and they redistribute the candies evenly again. This time, there are three left over. If the bag contains over 500 candies, what is the fewest number of candies the bag can contain?

2. [**3**] How many ways can 2^{2012} be expressed as the sum of four (not necessarily distinct) positive squares?

3. [**4**] Let the sequence $\{x_n\}$ be defined by $x_1 \in \{5, 7\}$ and, for $k \geq 1$, $x_{k+1} \in \{5^{x_k}, 7^{x_k}\}$. For example, the possible values of x_3 are 5^{5^5}, 5^{5^7}, 5^{7^5}, 5^{7^7}, 7^{5^5}, 7^{5^7}, 7^{7^5}, and 7^{7^7}. Determine the sum of all possible values for the last two digits of x_{2012}.

4. [**4**] Find the sum of all possible sums $a + b$ where a and b are nonnegative integers such that $4^a + 2^b + 5$ is a perfect square.

5. [**5**] Call a positive integer x a leader if there exists a positive integer n such that the decimal representation of x^n starts (not ends) with 2012. For example, 586 is a leader since $586^3 = 201230056$. How many leaders are there in the set $\{1, 2, 3, ..., 2012\}$?

6. [**6**] Let $p_1 = 2012$ and $p_n = 2012^{p_{n-1}}$ for $n > 1$. Find the largest integer k such that $p_{2012} - p_{2011}$ is divisible by 2011^k.

7. [**7**] Let a, b, and c be positive integers satisfying

$$a^4 + a^2b^2 + b^4 = 9633$$
$$2a^2 + a^2b^2 + 2b^2 + c^5 = 3605.$$

What is the sum of all distinct values of $a + b + c$?

8. [**8**] Find the largest possible sum $m + n$ for positive integers $m, n \leq 100$ such that $m + 1 \equiv 3 \pmod 4$ and there exists a prime number p and nonnegative integer a such $\frac{m^{2^n-1}-1}{m-1} = m^n + p^a$.