Number Theory A Solutions

Written by Albert Zhou

1. [**3**] Albert has a very large bag of candies and he wants to share all of it with his friends. At first, he splits the candies evenly amongst his 20 friends and himself and he finds that there are five left over. Ante arrives, and they redistribute the candies evenly again. This time, there are three left over. If the bag contains over 500 candies, what is the fewest number of candies the bag can contain?

   Solution: Note that 21 and 22 are relatively prime, so we can apply the Chinese Remainder Theorem to find that there is a unique solution modulo 462. Taking x to be the answer, we have

   $$x \equiv 5 \pmod{21}$$
   $$x \equiv 3 \pmod{22}$$

   Solving this system, we find that

   $$x \equiv 47 \pmod{462},$$

   so we have $x = \boxed{\mathbf{509}}$.

2. [**3**] How many ways can 2^{2012} be expressed as the sum of four (not necessarily distinct) positive squares?

   Solution: We have the equation $a^2 + b^2 + c^2 + d^2 = 2^{2012}$. First, consider the problem modulo 4. The only residues of squares modulo 4 are 0 and 1.

   If all of the squares have residues of 1 modulo 4, then they are all odd and we consider the problem modulo 8. The only residues of squares modulo 8 are 0, 1, and 4, and because $2^{2012} \equiv 0 \pmod 8$, we see that the squares cannot all be odd, so they must all be even.

   If all of the squares are even, then we divide both sides by 4 and repeat the process. We see that the only solution is

   $$a = b = c = d = 2^{1005},$$

   so there is only $\boxed{\mathbf{1}}$ solution.

   Problem contributed by Wesley Cao.

3. [**4**] Let the sequence $\{x_n\}$ be defined by $x_1 \in \{5, 7\}$ and, for $k \geq 1$, $x_{k+1} \in \{5^{x_k}, 7^{x_k}\}$. For example, the possible values of x_3 are 5^{5^5}, 5^{5^7}, 5^{7^5}, 5^{7^7}, 7^{5^5}, 7^{5^7}, 7^{7^5}, and 7^{7^7}. Determine the sum of all possible values for the last two digits of x_{2012}.

   Solution: Note that $7^4 = 2401 \equiv 1 \pmod{100}$ and that $5^n \equiv 25 (\bmod 100)$ for $n \geq 2$. Then we must consider 3 cases.

Case 1: We consider numbers of the form 5^x, where x is an odd positive integer greater than 1. Clearly, from our above observation, $5^x \equiv 25 \pmod{100}$.

Case 2: We consider numbers of the form 7^{5^x}, where x is an odd positive integer greater than 1. Then, we apply our observation to get $7^{5^x} \equiv 7^{25} \pmod{100}$, which we can further reduce to $7^{25} \equiv 7 \pmod{100}$.

Case 3: We consider numbers of the form 7^{7^x}, where x is an odd positive integer greater than 1. Then we have $7^x \equiv (-1)^x \pmod 4 \equiv -1 \pmod 4$. $7^3 \equiv 43 \pmod{100}$, so the residue is 43 in this case.

Finally, we have $25 + 7 + 43 = \boxed{\mathbf{75}}$.

Problem based on China 2010.

4. [**4**] Find the sum of all possible sums $a + b$ where a and b are nonnegative integers such that $4^a + 2^b + 5$ is a perfect square.

   Solution: This question is based on quadratic residues. If $a > 1$ and $b > 2$ then the resulting number is $5 \pmod 8$, hence not a perfect square.

   Then we check other cases 1 by 1:
   1) $b = 0$. This becomes $4^a + 6$ which is $2 \pmod 4$ $(a > 0)$ or $3 \pmod 4$ $(a = 0)$, not a perfect square.
   2) $a = 0$. This becomes $2^b + 6$ which is either $7, 8$, or is $2 \pmod 4$, so it's not a perfect square.
   3) $b = 1$. This becomes $4^a + 7$ which is either 8, or is $3 \pmod 4$, so it's not a perfect square.
   4) $a = 1$. This becomes $2 * b + 9$, which is $2 \pmod 3$ if b is odd (so it's not a perfect square), and if b is even, let $b = 2k$, then if $k > 2$ then $(2^k)^2 < 2^b + 9 < (2^k + 1)^2$, and only when $k = 2$ we get a perfect square. Hence only solution in this case is $a = 1, b = 4$.
   5) $b = 2$. This becomes $4^a + 9$, and if $a > 2$ then $(2^a)^2 < 4^a + 9 < (2^a + 1)^2$, and only when $a = 2$ we get a perfect square. Hence only solution in this case is $a = b = 2$.

   And so we get the solutions $a = 1, b = 4$ and $a = b = 2$. Thus, our answer is $\boxed{\mathbf{9}}$.

   Problem contributed by Chengyue Sun.

5. [**5**] Call a positive integer x a leader if there exists a positive integer n such that the decimal representation of x^n starts (not ends) with 2012. For example, 586 is a leader since $586^3 = 201230056$. How many leaders are there in the set $\{1, 2, 3, ..., 2012\}$?

   Solution: We see that x is a leader if and only if there exists a positive integer t such that

$$2.012 \times 10^s \leq x^n \leq 2.013 \times 10^s$$

   Because all values are greater than or equal to 1, we can take the logarithm of each part of the inequality, yielding

$$s + \log_{10} 2.012 \leq n \log_{10} x \leq s + \log_{10} 2.013.$$

   If $\log_{10} x$ is irrational, then we are guarenteed to find a n to satisfy these conditions, by the Equidistribution Theorem (this is also intuitively obvious). The only integers x for which

$\log_{10} x$ is rational are powers of 10. For $x = 1, 10, 100, 1000$, we can see that the leading digit is always 1, which means x is not a leader for these four numbers. All other numbers have an irrational common log, so there are $2012 - 4 = \boxed{\mathbf{2008}}$ leaders in the set.

Problem contributed by Wesley Cao.

6. [**6**] Let $p_1 = 2012$ and $p_n = 2012^{p_{n-1}}$ for $n > 1$. Find the largest integer k such that $p_{2012} - p_{2011}$ is divisible by 2011^k.

Solution: The difference in question is

$$p_{2012} - p_{2011} = p_{2011}\left((2012)^{p_{2011}-p_{2010}} - 1\right) = p_{2011}\left((2012)^{p_{2011}-p_{2010}} - 1^{p_{2011}-p_{2010}}\right).$$

We note that we can apply the Lifting the Exponent Lemma to the quantity in parentheses because 2011 is prime. The lemma states that if x and y are integers, n is a positive integer, and p is an odd prime such that $p|x - y$ but x and y are not divisible by p, we have

$$v_p(x^n - y^n) = v_p(x - y) + v_p(n)$$

where $v_p(m)$ refers the greatest power in which p divides m, i.e. $p^{v_p(m)}|m$ but $p^{v_p(m)+1} \nmid m$. So by this lemma,

$$v_{2011}\left((2012)^{p_{2011}-p_{2010}} - 1^{p_{2011}-p_{2010}}\right) = v_{2011}(2011) + v_{2011}(p_{2011} - p_{2010}),$$

where $v_{2011}(x)$ denotes the largest m such that 2011^m divides x. Clearly $v_{2011}(2011) = 1$, so we need to determine $v_{2011}(p_{2011} - p_{2010})$. But we note that this sequence is recursive, with 1 being added at each step. So we just need to find $v_{2011}(p_2 - p_1)$.

$$v_{2011}(p_2 - p_1) = v_{2011}\left(2012(2012^{2011} - 1^{2011}\right) = v_{2011}(2011) + v_{2011}(2011).$$

So $v_{2011}(p_2 - p_1) = 2$ and we have $v_{2011}(p_{2012} - p_{2011}) = 2012$, so $k = \boxed{\mathbf{2012}}$.
Problem contributed by Wesley Cao.

7. [**7**] Let a, b, and c be positive integers satisfying

$$a^4 + a^2b^2 + b^4 = 9633$$
$$2a^2 + a^2b^2 + 2b^2 + c^5 = 3605.$$

What is the sum of all distinct values of $a + b + c$?

Solution: We begin by summing the two systems and adding 1 to each side to obtain

$$a^4 + 2a^2b^2 + b^4 + 2a^2 + 2b^2 + 1 + c^5 = 13239,$$

which we can rewrite as

$$(a^2 + b^2 + 1)^2 + c^5 = 13239.$$

Now we consider this system modulo 11, because the least common multiple of 2 and 5 is 10, and by Fermat's Little Theorem, we have $x^{10} \equiv 1 (\bmod 11)$ whenever x is not a multiple of 11. Thus, the only possible residues for $\left(c^5\right)^2$ are 0 and 1, so the possible residues for c^5 are 0, ± 1. The only possible residues for squares modulo 11 are 0, 1, 3, 4, 5, and 9. Considering the right-hand side modulo 11, we find that the residue is 6, so

$$c^5 \equiv 1 \,(\bmod\, 11),\ \left(a^2 + b^2 + 1\right)^2 \equiv 5 \,(\bmod\, 11).$$

Going back to the second given equation, we see that we only have to check for $c = 1, 2, 3, 4, 5$. But $2^5 \equiv -1 \,(\bmod\, 11)$, so we only consider $c = 1, 3, 4, 5$. Considering our summed equation now, we see that c cannot be 1 because the last digit of a square cannot be 8. We can check that $c = 3$ is the only possible solution, with $a^2 + b^2 + 1 = 114$. The only solutions are $a = 7$, $b = 8$ and $a = 8$, $b = 7$, so $a + b + c = \boxed{\mathbf{18}}$.

8. [**8**] Find the largest possible sum $m + n$ for positive integers $m, n \leq 100$ such that $m + 1 \equiv 3 \,(\bmod\, 4)$ and there exists a prime number p and nonnegative integer a such $\frac{m^{2^n-1}-1}{m-1} = m^n + p^a$.

Solution: We consider two cases: $n = 2$ and $n > 2$. When $n = 2$, then

$$\frac{m^{2^n-1} - 1}{m - 1} = m^2 + m + 1.$$

Let $p = m + 1$, $a = 1$, and we are done.

For $n \geq 3$, let $n + 1 = 2^k q$, with $k \in \mathbf{N}$ and $q \in \mathbf{Z}^+$ and $2 \not| q$. Because

$$2^n = (1 + 1)^n \geq 1 + n + \frac{n(n-1)}{2} > n + 1,$$

we have $0 \leq k \leq n - 1$. But

$$\frac{m^{2^n} - 1}{m - 1} = \prod_{t=0}^{n-1} (m^{2^t} + 1),$$

so we have

$$m^{2^k} + 1 \mid \frac{m^{2^n} - 1}{m - 1},$$

$$m^{2^k} + 1 \mid m^{n+1} + 1 (= (m^{2^k})^q + 1).$$

Let $d_n = \frac{m^{2^n-1}-1}{m-1} - m^n$. Then we get

$$md_n = \frac{m^{2^n} - m}{m-1} - m^{n+1} = \frac{m^{2^n} - 1}{m-1} - (m^{n+1}+1).$$

Therefore $m^{2^k}+1 \mid md_n$, and from this we have $m^{2^k}+1 \mid d_n$. If $d_n = p^a$, then $p \mid m^{2^k}+1$, and from $2 \mid m$ and thus $2 \not| p$. Because

$$m^{p-1} \equiv 1 \pmod p,$$
$$m^{2^{k+1}} \equiv (-1)^2 \equiv 1 \pmod p,$$

we have

$$m^{(p-1,\,1^{k+1})} \equiv 1 \pmod p,$$

but

$$m^{2^k} \equiv -1 \not\equiv 1 \pmod p,$$

therefore $(p-1, 2^{k+1}) = 2^{k+1}$, so we have

$$p \equiv 1 \pmod{2^{k+1}}. \tag{1}$$

Note that

$$p^a = \frac{m^{2^n-1}-1}{m-1} - m^n = \sum_{t=0}^{2^n-2} m^t - m^n \equiv 1 + m + m^2 \pmod{m^3}. \tag{2}$$

If $k > 0$, then from (1) we know that $p \equiv 1 \pmod 4$, thus

$$p^a \equiv 1 \not\equiv 1 + m \pmod 4,$$

which is contrary to (2). Therefore, $k = 0$. From $p \mid m^{2^k}+1$, we have $p \mid m+1$, and because $m+1$ is a prime number, we have $p = m+1$, so $p^a \equiv 1 \pmod 8$ or $p^a = m+1 \pmod 8$, but $1+m+m^2 \not\equiv 1,\ m+1 \pmod 8$, which contradicts (2). Therefore, there are no solutions for $n \geq 3$, so the only m, n satisfying the conditions are: $n = 2$, $m = q-1$, where q is any prime number such that $q \equiv 3 \pmod 4$.

All that is left is to determine the largest prime number less than 100 that has a residue of 3 modulo 4. This number is 83, so $m = 82$ and $m + n = \boxed{\mathbf{84}}$.

Problem based on China 2010.