PUMaC 2013

Algebra A

1. [**3**] Suppose $a, b, c > 0$ are integers such that

$$abc - bc - ac - ab + a + b + c = 2013.$$

Find the number of possibilities for the ordered triple (a, b, c).

2. [**3**] Find the number of pairs (n, C) of positive integers such that $C \leq 100$ and $n^2 + n + C$ is a perfect square.

3. [**4**] Let $x_1 = \sqrt{10}$ and $y_1 = \sqrt{3}$. For all $n \geq 2$, let

$$x_n = x_{n-1}\sqrt{77} + 15y_{n-1}$$
$$y_n = 5x_{n-1} + y_{n-1}\sqrt{77}$$

Find $x_5^6 + 2x_5^4 - 9x_5^4y_5^2 - 12x_5^2y_5^2 + 27x_5^2y_5^4 + 18y_5^4 - 27y_5^6$.

4. [**4**] Suppose a, b are nonzero integers such that two roots of $x^3 + ax^2 + bx + 9a$ coincide, and all three roots are integers. Find $|ab|$.

5. [**5**] Suppose w, x, y, z satisfy

$$w + x + y + z = 25$$
$$wx + wy + wz + xy + xz + yz = 2y + 2z + 193$$

The largest possible value of w can be expressed in lowest terms as w_1/w_2 for some integers $w_1, w_2 > 0$. Find $w_1 + w_2$.

6. [**6**] Suppose the function ψ satisfies $\psi(1) = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$ and $\psi(3x) + 3\psi(x) = \psi(x)^3$ for all real x. Determine the greatest integer less than $\prod_{n=1}^{100} \psi(3^n)$.

7. [**7**] Evaluate

$$\sqrt{2013 + 276\sqrt{2027 + 278\sqrt{2041 + 280\sqrt{2055 + \ldots}}}}$$

8. [**8**] Let $\mathcal{S}$ be the set of permutations of $\{1, 2, \ldots, 6\}$, and let $\mathcal{T}$ be the set of permutations of $\mathcal{S}$ that preserve compositions: i.e., if $F \in \mathcal{T}$, then

$$F(f_2 \circ f_1) = F(f_2) \circ F(f_1).$$

for all $f_1, f_2 \in \mathcal{S}$. Find the number of elements $F \in \mathcal{T}$ such that if $f \in \mathcal{S}$ satisfies $f(1) = 2$ and $f(2) = 1$, then $(F(f))(1) = 2$ and $(F(f))(2) = 1$.