PUMaC 2013

Combinatorics A

1. [**3**] A regular pentagon can have the line segments forming its boundary extended to lines, giving an arrangement of lines that intersect at ten points. How many ways are there to choose five points of these ten so that no three of the points are colinear?

2. [**3**] How many ways are there to color the edges of a hexagon orange and black if we assume that two hexagons are indistinguishable if one can be rotated into the other? Note that we are saying the colorings OOBBOB and BOBBOO are distinct; we ignore flips.

3. [**4**] How many tuples of integers $(a_0, a_1, a_2, a_3, a_4)$ are there, with $1 \leq a_i \leq 5$ for each i, so that $a_0 < a_1 > a_2 < a_3 > a_4$?

4. [**4**] You roll three fair six-sided dice. Given that the highest number you rolled is a 5, the expected value of the sum of the three dice can be written as $\frac{a}{b}$ in simplest form. Find $a + b$.

5. [**5**] Mereduth has many red boxes and many blue boxes. Coloon has placed five green boxes in a row on the ground, and Mereduth wants to arrange some number of her boxes on top of his row. Assume that each box must be placed so that it straddles two lower boxes. Including the one with no boxes, how many arrangements can Mereduth make?

6. [**6**] A sequence of vertices $v_1, v_2, \ldots, v_k$ in a graph, where $v_i = v_j$ only if $i = j$ and k can be any positive integer, is called a *cycle* if v_1 is attached by an edge to v_2, v_2 to v_3, and so on to v_k connected to v_1. Rotations and reflections are distinct: A, B, C is distinct from A, C, B and B, C, A. Suppose a simple graph G has 2013 vertices and 3013 edges. What is the minimal number of cycles possible in G?

7. [**7**] The Miami Heat and the San Antonio Spurs are playing a best-of-five series basketball championship, in which the team that first wins three games wins the whole series. Assume that the probability that the Heat wins a given game is x (there are no ties). The expected value for the total number of games played can be written as $f(x)$, with f a polynomial. Find $f(-1)$.

8. [**8**] Eight all different sushis are placed evenly on the edge of a round table, whose surface can rotate around the center. Eight people also evenly sit around the table, each with one sushi in front. Each person has one favorite sushi among these eight, and they are all distinct. They find that no matter how they rotate the table, there are never more than three people who have their favorite sushis in front of them simultaneously. By this requirement, how many different possible arrangements of the eight sushis are there? Two arrangements that differ by a rotation are considered the same.