PUMaC 2013

Combinatorics A Solutions

1. [**3**] A regular pentagon can have the line segments forming its boundary extended to lines, giving an arrangement of lines that intersect at ten points. How many ways are there to choose five points of these ten so that no three of the points are colinear?

   Solution There are five lines and five points, each point straddling two lines. Since no line contains more than two points, then, each line must contain exactly two points. Considering each point as an intersection of two lines in the set $\{a, b, c, d, e\}$ of lines, the points can be expressed as a cycle of lines: eg, $\{ab, bd, de, ec, ca\}$ or $\{a, b, d, e, c\}$. Two cycles correspond to the same set of points only when they are a rotation or reflection of another cycle, as two points are the same, $ab = cd$, only if $a = c, b = d$ or $a = d, b = c$. There are $4! = 24$ cycles for 12 arrangements.

2. [**3**] How many ways are there to color the edges of a hexagon orange and black if we assume that two hexagons are indistinguishable if one can be rotated into the other? Note that we are saying the colorings OOBBOB and BOBBOO are distinct; we ignore flips.

   Solution The rotation group for the hexagon consists of the identity, two rotate-by-$\pi/3$ operations, two rotate-by-$2\pi/3$, and one rotate by π operation. There are 64 colorings fixed by the identity, 2 by the two rotate-by-$\pi/3$ operations, 4 by the two rotate-by-$2\pi/3$ operations, and 8 by the rotate-by-π operation. Then the number of distinct colorings is given by Burnside's lemma to be $\frac{1}{6}(1(64) + 2(2) + 2(4) + 1(8)) = \frac{84}{6} = 14$. So there are 14 colorings.

3. [**4**] How many tuples of integers $(a_0, a_1, a_2, a_3, a_4)$ are there, with $1 \leq a_i \leq 5$ for each i, so that $a_0 < a_1 > a_2 < a_3 > a_4$?

   Solution Denote by $S(n, k)$ the number of tuples $a_0 < a_1 > \dots a_n$ so $0 \leq a_1, \dots, a_{n-1} \leq 5$ and $a_n = k$. Then we have $S(0, 1) = S(0, 2) = S(0, 3) = S(0, 4) = S(0, 5)$. We next find that, if $a_0 < k$, either $a_0 < k - 1$ or $a_0 = k - 1$, giving

   $$S(1, k) = S(1, k - 1) + S(0, k - 1)$$

   so $S(1, 1) = 0$, $S(1, 2) = 1$, $S(1, 3) = 2$, $S(1, 4) = 3$, $S(1, 5) = 4$. In general, we find

   $$S(2n + 1, k) = S(2n + 1, k - 1) + S(2n, k - 1)$$

   $$S(2n, k) = S(2n, k + 1) + S(2n - 1, k + 1)$$

   Then

   $S(2, 5) = 0$, $S(2, 4) = 4$, $S(2, 3) = 7$, $S(2, 2) = 9$, $S(2, 1) = 10$

   $S(3, 1) = 0$, $S(3, 2) = 10$, $S(3, 3) = 19$, $S(3, 4) = 26$, $S(3, 5) = 30$

   $S(4, 5) = 0$, $S(4, 4) = 30$, $S(4, 3) = 56$, $S(4, 2) = 75$, $S(4, 1) = 85$.

   Summing the bottom row gives us the answer of $30 + 56 + 75 + 85 = 246$ tuples.

4. [**4**] You roll three fair six-sided dice. Given that the highest number you rolled is a 5, the expected value of the sum of the three dice can be written as $\frac{a}{b}$ in simplest form. Find $a + b$.

Solution Note that this is *not* the same as fixing one die at 5, then randomizing the rest from 1 to 5. To compute the expected value correctly, we consider three cases: rolls that include exactly one, two, three 5s. To make counting easier we think of the rolls as happening sequentially and count each distinct sequence as an instance. Non-5 rolls are equally likely to be 1, 2, 3, 4 and are therefore worth 2.5 on average. So we have:

Three 5s: $(1 * 4^0) * 15 = 15$

Two 5s: $(3 * 4^1) * 12.5 = 150$

One 5: $(3 * 4^2) * 10.0 = 480$

We end up with $1 + 12 + 48 = 61$ instances and a total sum of 645. The fraction $\frac{645}{61}$, already in simplest form, gives sum 706, and that is the answer.

5. [**5**] Mereduth has many red boxes and many blue boxes. Coloon has placed five green boxes in a row on the ground, and Mereduth wants to arrange some number of her boxes on top of his row. Assume that each box must be placed so that it straddles two lower boxes. Including the one with no boxes, how many arrangements can Mereduth make?

   Solution Let K_n be the number of arrangements we can make on n boxes. On top of the n green boxes, Mereduth has some string of boxes with no gaps, possibly of length zero. If this is of length i, we find that the number of further arrangements that can be made is $2^i K_i K_{n-i-1}$. Then we have $K_0 = 1$, $K_1 = 1$, $K_2 = 2K_1K_0 + K_0K_1 = 3$, $K_3 = 4K_2K_0 + 2K_1^2 + K_0K_2 = 4(3) + 2 + 3 = 17$, $K_4 = 8K_3K_0 + 4K_2K_1 + 2K_1K_2 + K_0K_3 = 9(17) + 6(3) = 171$, and

$$K_5 = 17 \times 171 + 10 \times 17 + 4 \times 3^2 = 2907 + 160 + 36 = 3113$$

   So there are 3113 rearrangements.

6. [**6**] A sequence of vertices $v_1, v_2, \ldots, v_k$ in a graph, where $v_i = v_j$ only if $i = j$ and k can be any positive integer, is called a *cycle* if v_1 is attached by an edge to v_2, v_2 to v_3, and so on to v_k connected to v_1. Rotations and reflections are distinct: A, B, C is distinct from A, C, B and B, C, A. Suppose a simple graph G has 2013 vertices and 3013 edges. What is the minimal number of cycles possible in G?

   Solution We assume G is connected. Then 2012 edges make a spanning subtree of G. Adding any edge is going to allow a new loop; if this loop involves n vertices, $2n$ cycles result. Each new edge is going to be involved in at least six new cycles, then, and it can be seen pretty easily that this bound is attained. So the minimum is 6006.

7. [**7**] The Miami Heat and the San Antonio Spurs are playing a best-of-five series basketball championship, in which the team that first wins three games wins the whole series. Assume that the probability that the Heat wins a given game is x (there are no ties). The expected value for the total number of games played can be written as $f(x)$, with f a polynomial. Find $f(-1)$.

   Solution The chance of the Heat winning in three is x^3 and the chance of them losing in three is $(1-x)^3$. The chance of them winning in four is $3x^3(1-x)$, and the chance of losing is $3x(1-x)^3$. The chance of them winning in five is $6x^3(1-x)^2$, losing is $6x^2(1-x)^3$. In short,

$$f(x) = 3(x^3 + (1-x)^3) + 4(3)(x^3(1-x) + x(1-x)^3) + 5(6)(x^3(1-x)^2 + x^2(1-x)^3)$$

Subbing in $x = -1$ gives

$$f(-1) = 3(-1+8) + 12(-2+-8) + 30(-4+8) = 21 - 120 + 120 = 21$$

So the answer is 21.

8. [**8**] Eight all different sushis are placed evenly on the edge of a round table, whose surface can rotate around the center. Eight people also evenly sit around the table, each with one sushi in front. Each person has one favorite sushi among these eight, and they are all distinct. They find that no matter how they rotate the table, there are never more than three people who have their favorite sushis in front of them simultaneously. By this requirement, how many different possible arrangements of the eight sushis are there? Two arrangements that differ by a rotation are considered the same.

   Solution Under rotation, there are totally $\frac{8!}{8}$ different arrangements. We find the answer by subtracting the cases where more than 3 people can match their favorite sushi by rotation. Take an example of the case where maximum exactly 4 people can match their favorite sushi by some rotation. (Cases of more than 4 people are even simpler.)

   Suppose we have already rotated the table to the optimal position, when 4 people get matched. There are $\binom{8}{4}$ ways of choosing these 4 people. The other 4 people must be all mismatched at this moment. This is the number of "Error Permutation" and can be calculated with recursive relation $f(n) = (n-1)(f(n-1) + f(n-2))$, or simply by enumeration. It turns out to be 9. But some arrangements are counted twice. These are the cases when the mismatched 4 people can all get matched by another rotation.

   If we number the sushis from 1 to 8 by the order of positions around the table, suppose 4 people are matched to their favorite sushi at one moment, and (a, b, c, d) $(a < b < c < d)$ are the ones that are mismatched. If under some rotation, all of (a, b, c, d) get matched, then originally the favorite sushi of the people in front of (a, b, c, d) must be (b, c, d, a), (c, d, a, b) or (d, a, b, c). So either

$$(b-a) \equiv (c-b) \equiv (d-c) \equiv (a-d) \pmod 8$$

   or

$$\begin{cases} (b-a) \equiv (d-c) & \pmod 8 \\ (c-b) \equiv (a-d) & \pmod 8 \\ (b-a) \not\equiv (c-b) & \pmod 8 \end{cases}$$

   In the first case, $(a, b, c, d) = (1, 3, 5, 7)$ or $(2, 4, 6, 8)$, and 3 arrangements are counted twice.

   In the second case, $(a, b, c, d) = (1, 2, 5, 6)$ or $(2, 3, 6, 7)$ or $(3, 4, 7, 8)$ or $(1, 4, 5, 8)$, and 2 arrangements are counted twice. So in total there are 5 extra counts.

   The answer is

$$\frac{8!}{8} - \binom{8}{8} - \binom{8}{6} - \binom{8}{5} \times 2 - \binom{8}{4} \times 9 + 5 = \boxed{4274}$$