PUMaC 2013

Geometry A

1. [**3**] Let O be a point with three other points A, B, C and $\angle AOB = \angle BOC = \angle AOC = 2\pi/3$. Consider the average area of the set of triangles ABC where $OA, OB, OC \in \{3, 4, 5\}$. The average area can be written in the form $m\sqrt{n}$ where m, n are integers and n is not divisible by a perfect square greater than 1. Find $m + n$.

2. [**3**] An equilateral triangle is given. A point lies on the incircle of the triangle. If the smallest two distances from the point to the sides of the triangle is 1 and 4, the sidelength of this equilateral triangle can be expressed as $\frac{a\sqrt{b}}{c}$ where $(a, c) = 1$ and b is not divisible by the square of an integer greater than 1. Find $a + b + c$.

3. [**4**] Consider the shape formed from taking equilateral triangle ABC with side length 6 and tracing out the arc BC with center A. Set the shape down on line l so that segment AB is perpendicular to l, and B touches l. Beginning from arc BC touching l, We roll ABC along l until both points A and C are on the line. The area traced out by the roll can be written in the form $n\pi$, where n is an integer. Find n.

4. [**4**] Draw an equilateral triangle with center O. Rotate the equilateral triangle $30°, 60°, 90°$ with respect to O so there would be four congruent equilateral triangles on each other. Look at the diagram. If the smallest triangle has area 1, the area of the original equilateral triangle could be expressed as $p + q\sqrt{r}$ where p, q, r are positive integers and r is not divisible by a square greater than 1. Find $p + q + r$.

5. [**5**] Suppose you have a sphere tangent to xy-plane with its center having positive z-coordinate. If it is projected from a point $P = (0, b, a)$ to the xy-plane, it gives the conic section $y = x^2$. If we write $a = \frac{p}{q}$ where p, q are integers, find $p + q$.

6. [**6**] On a circle, points A, B, C, D lie counterclockwise in this order. Let the orthocenters of ABC, BCD, CDA, DAB be H, I, J, K respectively. Let $HI = 2, IJ = 3, JK = 4, KH = 5$. Find the value of $13(BD)^2$.

7. [**7**] Given triangle ABC and a point P inside it, $\angle BAP = 18°$, $\angle CAP = 30°$, $\angle ACP = 48°$, and $AP = BC$. If $\angle BCP = x°$, find x.

8. [**8**] Three chords of a sphere, each having length $5, 6, 7$, intersect at a single point inside the sphere and are pairwise perpendicular. For R the minimum possible radius of the sphere, find R^2.