Number Theory A Solutions

1. [**3**] If p, q and r are primes with $pqr = 7(p+q+r)$, find $p+q+r$.

   Solution Without loss of generality, we see that we must have $p = 7$. Next, $qr = 7+q+r$ iff $(q-1)(r-1) = 8$. The only prime solution is $(q, r) = (3, 5)$ up to permutation. Then $p+q+r = 15$.

2. [**3**] What is the smallest positive integer n such that 2013^n ends in 001 (i.e. the rightmost three digits of 2013^n are 001)?

   Solution Firstly,

$$\begin{aligned} 2013^{100} &\equiv 13^{100} \\ &= (10+3)^{100} \\ &\equiv 3^{100} \\ &\equiv 49^{10} \\ &\equiv 401^5 \\ &= (400+1)^5 \\ &\equiv 1 \pmod{1000} \end{aligned}$$

   yields $n|100$. Next

$$\begin{aligned} 2013^{50} &\equiv (10+3)^{50} \\ &\equiv 3^{50} + 50 \cdot 10 \cdot 3^{49} + \binom{50}{2} \cdot 10^2 \cdot 3^{48} \\ &\equiv 3^{50} \\ &\equiv 49^5 \\ &\equiv 807^2 \\ &\equiv 249 \pmod{1000} \end{aligned}$$

   and

$$\begin{aligned} 2013^{20} &\equiv (10+3)^{20} \\ &\equiv 3^{20} + 20 \cdot 10 \cdot 3^{19} \\ &\equiv 49^2 + 400 \\ &\equiv 801 \pmod{1000} \end{aligned}$$

   shows that $n = 100$.

3. [**4**] Let A be the greatest possible value of a product of positive integers that sums to 2014. Compute the sum of all bases and exponents in the prime factorization of A. For example, if $A = 7 \cdot 11^5$, the answer would be $7 + 11 + 5 = 23$.

Solution Note that if we have a large enough n, odd $n \geq 5$, break it as $n = \frac{n-1}{2} + \frac{n+1}{2}$. This product is larger than n For even $n \geq 4$, break it as $n = \frac{n}{2} + \frac{n}{2}$. This product is larger than n. Furthermore, noting that $2^3 < 3^2$, we should have at most two 2's. So the optimum case is $(2)(2) + (670)(3) = 2014$. The answer is 677.

4. [**4**] Let d be the greatest common divisor of $2^{30^{10}} - 2$ and $2^{30^{45}} - 2$. Find the remainder when d is divided by 2013.

   Solution We have

$$d = (2^{30^{10}} - 2, 2^{30^{45}} - 2) = 2 \cdot (2^{(30^{10}-1, 30^{45}-1)} - 1) = 2 \cdot (2^{30^{(10,45)}-1} - 1) = 2^{30^5} - 2.$$

   As $\phi(2013) = 1200$ and $1200 | 30^5$, the remainder is 2012.

5. [**5**] Define a "digitized number" as a ten-digit number $a_0 a_1 \ldots a_9$ such that for $k = 0, 1, \ldots, 9$, a_k is equal to the number of times the digit k occurs in the number. Find the sum of all digitized numbers.

   Solution From the condition, we need

$$\sum_{k=0}^{9} k a_k = \sum_{k=0}^{9} a_k = 10.$$

   In particular,

$$a_0 = \sum_{k=2}^{9} (k-1) a_k \geq \sum_{k=2}^{9-a_0} (k-1)$$

   yields $a_0 \geq 6$. By examining each case, we find the only solution 6210001000.

6. [**6**] What is the largest positive integer that cannot be expressed as a sum of non-negative integer multiples of 13, 17 and 23?

   Solution There are numerous approaches to this problem, and no approach that attempts to find the last obtained remainder modulo any of the three numbers in sums of them will fail. The below is our approach:
   By trial, the following gives the smallest positive integer in the form $17a + 23b$ for integers $a, b \geq 0$ in different residue classes modulo 13 (starting from the class 0):

$$91, 40, 80, 68, 17, 57, 97, 46, 34, 74, 23, 63, 51.$$

   (Indeed, one should note that $5(17) \equiv 2(23) \pmod{13}$ and $3(23) \equiv 17 \pmod{13}$ to reduce the number of cases for consideration.)
   The largest n which is not of the form $13x + 17y + 23z$ is therefore $97 - 13 = 84$.

7. [**7**] Suppose $P(x)$ is a degree n monic polynomial with integer coefficients such that 2013 divides $P(r)$ for exactly 1000 values of r between 1 and 2013 inclusive. Find the minimum value of n.

   Solution Let a, b, and c be the number of solutions to the congruence equations $P(x) \equiv 0 \pmod{3}$,
   $P(x) \equiv 0 \pmod{11}$, $P(x) \equiv 0 \pmod{61}$ respectively.
   Then $abc = 1000$ and $a \leq \min\{3, n\}$, $b \leq \min\{11, n\}$, $c \leq \min\{61, n\}$. Then $\max a = 2$ and $\max b = 10$ so that $\min c = 50$. In particular, $\min n = 50$. The existence of such P is clear.

8. [**8**] Find the number of primes p between 100 and 200 for which $x^{11} + y^{16} \equiv 2013 \pmod{p}$ has a solution in integers x and y.

   Solution Note that if $p \neq 1 \pmod{11}$, then x^{11} will cycle over all residues modulo p, and thus there will always be a solution.
   It is thus left to check the primes that are 1 modulo 11.
   Of the numbers between 100 and 200 that are prime, 199 is the only one that is $1 \pmod{11}$.
   It remains to conclude that there is a solution to this equation mod 199. For example, one solution is $x = 5, y = 46$.
   Therefore the answer is simply the number of primes between 100 and 200, which is 21.