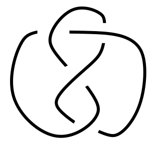
# PUMaC 2013 Power Round



## Rules, Remarks, and Reminders

These rules supersede any rules appearing elsewhere about the Power Round:

1. Your solutions are to be turned in when your team checks in on the morning of PUMaC. Please staple your solutions together, include the Power Round cover sheet as the first page, and write your team name on every sheet of paper you turn in.

2. On any problem, you may use any "Fact" or "Remark" in the Power Round. You may use without proof the result of any problem from earlier in the test, even if it's a problem your team has not solved. You may not cite results from conjectures or subsequent problems unless your team solved them independently of the problem where you wish to cite them.

3. It is not necessary to do the problems in order, although it is a good idea to read all the problems, so that you know what is permissible to assume when doing each problem. However, please collate the solutions in order in your solution packet.

4. Using calculators and Mathematica (or similar programs), is allowed. **Print and online sources are not allowed. No communication with humans outside your team about the content of these problems is allowed.**

5. For your convenience, we have provided both a table of contents and an index of terms and notation.

6. The first problem is 2.1.1. (In general, the problems are numbered *a.b.c.*) Point values for each problem are displayed in parenthesis next to the problem number.

7. Have any questions regarding the test? Please contact us at pumac@math.princeton.edu.

Good luck and have fun!

–Alan Chang ◡̈

# Contents

# 1 Preliminaries

## 1.1 Definitions

Although there are more technical definitions, for this Power Round, it is enough to think of a *knot* as something made physically by attaching the two ends of a string together. Since knots exist in three dimensions, when we need to draw them on paper, we often use *knot diagrams*. Figure 1.1 contains examples of knot diagrams.

(a) unknot (b) trefoil (c) trefoil (again) (d) figure-8 knot

Figure 1.1: Examples of knot diagrams, with the names of the knots they represent.

As we can see from Figure 1.1b and Figure 1.1c, different knot diagrams can represent the same knot. To see that these two are really the same knot, we could make Figure 1.1b out of a piece of string and move the string around in space (without cutting it) so that it looks like Figure 1.1c.

**Tip 1.1.** Make sure you understand the distinction between "knot" and "knot diagram"! Do *not* use these terms interchangeably in your solutions. If you use one when you mean the other, the proof will be incorrect (logically) and this will lead to a significantly lower score. ♢

There are some restrictions on knot diagrams: (1) each crossing must involve exactly two segments of the string and (2) those segments must cross transversely. (See Figure 1.2.)

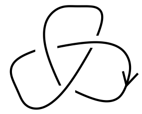
(a) triple crossing (b) non-transverse crossing

Figure 1.2: Examples of invalid knot diagrams

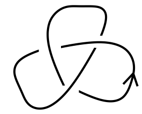
There are two ways to travel around a knot; these correspond to the *orientations* of the knot. An *oriented* knot is a knot with a specified orientation. On a knot diagram, we can indicate an orientation via an arrow. (See Figure 1.3.)

Sometimes we'll use more than one piece of string, so we define a *link* to be a generalization of a knot: links can be made by multiple pieces of string. For each string, we attach the two ends together. (Note that we do not attach the ends of two different strings together.)

The number of *components* of a link is the number of strings used. (Observe that every knot is a link with one component.) A *link diagram* is a straightforward generalization



(a) one orientation

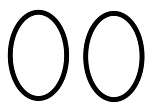


(b) the other orientation

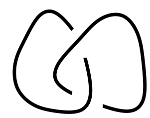
Figure 1.3: Two orientations of the trefoil

of a knot diagram, and an *oriented link* is a link where all the components have specified orientations.

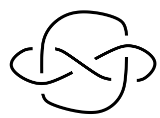
Figure 1.4 contains examples of two-component links.



(a) unlink



(b) Hopf link



(c) Whitehead link

Figure 1.4: Examples of links with two components

**Tip 1.2.** Pay close attention to the problem statements in this Power Round. If a problem asks you to prove something for links, it is not enough to prove it for knots! ◊

A *knot invariant* is something (such as number, matrix, or polynomial) associated to a knot. A *link invariant* is defined similarly for links. An example of a knot invariant is the *crossing number*, which we will now define.

Suppose for a knot $K$, we take a diagram $D$ of $K$ and count the number of crossings in $D$. This number is *not* an invariant of $K$ because $K$ has many different diagrams that differ in number of crossings. For example, in Figure 1.5, we see two different diagrams of the unknot.

(a) 0 crossings

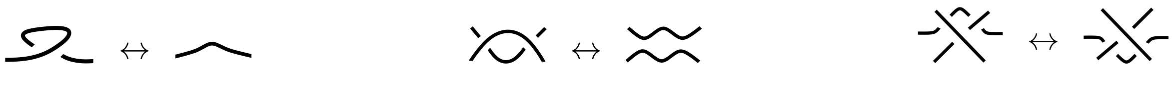
(b) 3 crossings

Figure 1.5: Two diagrams of the unknot, with different number of crossings.

Thus, we have not yet successfully defined a knot invariant. However, if we consider *all* diagrams of $K$ and take the *minimum* number of crossings over all diagrams, then we do have an invariant of $K$. This is called the *crossing number* of a knot.

## 1.2 Reidemeister moves

Consider the kinds of moves in Figure 1.6, which you can perform on a knot diagram. These are called *Reidemeister moves*. We can think of Type I as adding/removing a twist, Type II as crossing/uncrossing two strands, and Type III as sliding a strand past a crossing.



(a) Type I (b) Type II (c) Type III

Figure 1.6: The three types of Reidemeister moves

**Fact 1.3.** Suppose we start with a knot diagram $D_1$ and perform one of the Reidemeister moves on it so that we end up with a knot diagram $D_2$. The knot diagrams $D_1$ and $D_2$ might be different, but they will represent the same knot. (This is clear from the diagrams in Figure 1.6.) ♢

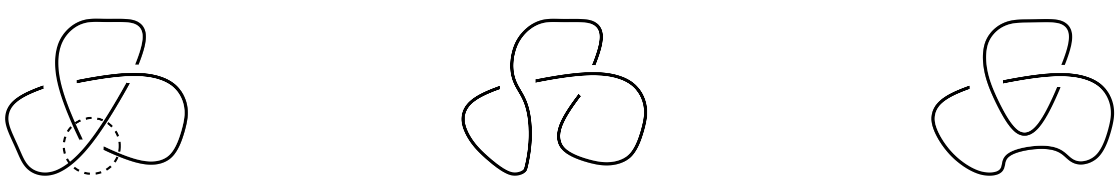
**Fact 1.4.** In 1926, Kurt Reidemeister proved that given two diagrams $D_1$ and $D_2$ of the same link, it is always possible to get from $D_1$ to $D_2$ via a finite sequence of Reidemeister moves. This is a remarkable fact! ♢

**Remark 1.5.** Suppose there is a quantity that we are trying to show is a knot invariant, but it is defined in terms of a knot diagram. There is a possibility that the quantity is different for different diagrams of the same knot, in which case our quantity would not be a knot invariant. However, if we can show the quantity is unchanged when we alter the diagram via any Reidemeister move, then we know it is an invariant, because of Fact 1.4. ♢

# 2 The Jones Polynomial

## 2.1 Resolving a crossing (4 points)

**Definition 2.1.** Suppose we start with a crossing of the form . The 0-resolution of this crossing is  and the 1-resolution is . (For example, see Figure 2.1.) ♢



(a) trefoil (b) 0-resolution (c) 1-resolution

Figure 2.1: Resolving a crossing

**Remark 2.2.** A diagram may need to be rotated so that the crossing in concern appears as ╳. For example, after a 90° rotation, we see that the 0- and 1-resolutions of ╳ are ≍ and )(, respectively. ◊

**Remark 2.3.** Here is one way to think of a 0-resolution: if we are traveling along a knot and reach a crossing in which we are on the upper strand, then we turn left onto the lower strand. (For a 1-resolution, we would turn right instead.) ◊

2.1.1. **(2)** Start with the diagram of the figure 8 knot given in Figure 2.2. Observe that the crossings are labeled $A$, $B$, $C$, $D$. Draw the diagram in which crossing $B$ has been resolved with a 0-resolution and identify the resulting knot/link.

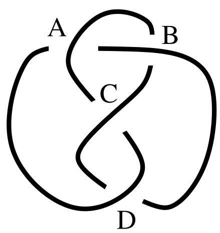



Figure 2.2

2.1.2. **(2)** Once again, start with the diagram of the figure 8 knot given in Figure 2.2. It is possible to resolve one of the crossings so that the resulting diagram is a trefoil. Which crossing can we resolve ($A$, $B$, $C$, $D$) and how do we resolve it (0- or 1-resolution)? Simply state an answer.

## 2.2 The Bracket Polynomial (0 points)

**Definition 2.4.** As you all know, a polynomial in $x$ is something of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

A *Laurent polynomial* in $x$ is like a polynomial except you can use negative powers. In other words a Laurent polynomial is something of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_m x^m,$$

where $m$ and $n$ are integers with $m \leq n$. (For example, $x^3 + 2x + 4 + x^{-1} - 5x^{-4}$ is a Laurent polynomial.) Quite confusingly, a Laurent polynomial is not necessarily a polynomial. ◊

**Definition 2.5.** The *bracket polynomial* of a link diagram $D$ is a Laurent polynomial in the variable $A$ and is denoted $\langle D \rangle$. It is completely determined by three rules:

$$\langle \bigcirc \rangle = 1 \qquad \text{(BP1)}$$

$$\langle D \sqcup \bigcirc \rangle = (-A^2 - A^{-2}) \langle D \rangle \qquad \text{(BP2)}$$

$$\langle ╳ \rangle = A \langle )( \rangle + A^{-1} \langle ≍ \rangle \qquad \text{(BP3)}$$

(The BP stands for "bracket polynomial.") ◊

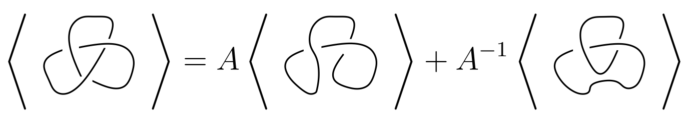
Let's go through what these rules mean, one by one.

1. The first relation (BP1) states that the bracket polynomial of the knot diagram $\bigcirc$ is the constant polynomial 1. (Note, however, that this does *not* mean that the bracket polynomial of *any* diagram depicting the unknot is 1. For example,  is also a diagram of the unknot, but this diagram turns out *not* to have bracket polynomial 1.)

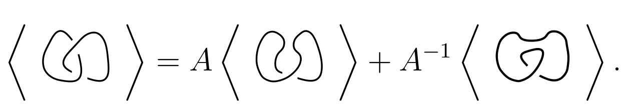
2. For the second relation, the expression $D \sqcup \bigcirc$ denotes a diagram $D$ with an extra circle added. Furthermore, the circle does not cross the rest of the diagram. If we do have a diagram of this form, then BP2 means that we can find its bracket polynomial by starting with the bracket polynomial of the diagram with the circle removed and multiplying it by $-A^2 - A^{-2}$. For example, using BP2 (along with BP1), we have

$$\langle \bigcirc\bigcirc \rangle = (-A^2 - A^{-2}) \langle \bigcirc \rangle = -A^2 - A^{-2}.$$

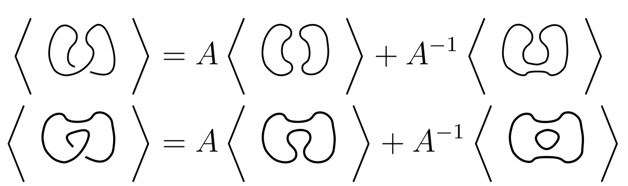
3. In order to apply the third relation, we need to resolve crossings. Start with a diagram $D$ and fix a crossing. If $D_0$ and $D_1$ are the 0- and 1-resolutions of this crossing, then BP3 states that $\langle D \rangle = A \langle D_0 \rangle + A^{-1} \langle D_1 \rangle$. For example,



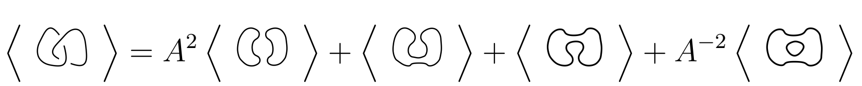
**Example 2.6.** Let's compute the bracket polynomial of the diagram . (It is a diagram of the Hopf link.) Applying BP3 gives us

 (2.1)

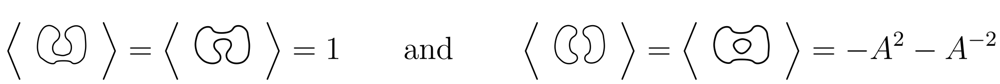
Using BP3 again,

 (2.2)

Combining (2.1) and (2.2), we see that

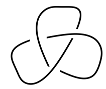


Invoking BP1 and BP2 gives us



Putting everything together gives us  $= -A^4 - A^{-4}$. ◇

**Fact 2.7.** For the trefoil, we have

$$\left\langle \right.$$  $$\left. \right\rangle = A^7 - A^3 - A^{-5}.$$

You might want to verify this yourself, to make sure you understand how to compute bracket polynomials of knot diagrams. $\Diamond$

## 2.3 Smoothings (10 points)

In Example 2.6, we decomposed the Hopf link into four diagrams. The four diagrams correspond to the four ways of resolving the two crossings of . Each of these diagrams is called a *smoothing*.

**Definition 2.8.** Given a link diagram $D$, a *smoothing* of $D$ is a diagram in which every crossing of $D$ has been resolved (either by a 0-resolution or a 1-resolution). Note that a smoothing has no crossings. $\Diamond$

**Definition 2.9.** Let $\{0,1\}^n$ denote the set of $n$-tuples, where each component is either 0 or 1. $\Diamond$

**Definition 2.10.** Let $D$ be a link diagram with $n$ crossings. Number the crossings $1, \dots, n$. Let $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{0,1\}^n$. Then $D_\epsilon$ denotes the smoothing of $D$ where crossing $i$ is resolved via a $\epsilon_i$-resolution for $i = 1, \dots, n$. (Note that $D_\epsilon$ is also a knot diagram.) $\Diamond$

**Definition 2.11.** Let $D$ be a link diagram, and let $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \{0,1\}^n$ be a smoothing of $D$. Define

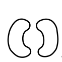
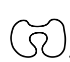
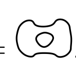
$$\begin{aligned} s_0(\epsilon) &= \text{the number of 0-resolutions in } D_\epsilon \\ s_1(\epsilon) &= \text{the number of 1-resolutions in } D_\epsilon \\ o(\epsilon) &= \text{the number of circles in } D_\epsilon. \end{aligned}$$

(We use the letter $o$ because it looks like a circle!) Also, define

$$\langle D, \epsilon \rangle = A^{s_0(\epsilon) - s_1(\epsilon)} \langle D_\epsilon \rangle. \qquad \Diamond$$

**Remark 2.12.** We will omit the commas in the $(\epsilon_1, \dots, \epsilon_n)$ notation to avoid clutter. For example, $\epsilon = 10011$ is short for $\epsilon = (1,0,0,1,1)$. $\Diamond$

**Example 2.13.** If $D =$ , and the top crossing is labeled 1 (so the bottom crossing is labeled 2), then

$D_{00} =$ , $D_{01} =$ , $D_{10} =$ , $D_{11} =$ . (2.3)

Also, $s_0(00) = 2$, $s_0(01) = s_0(10) = 1$, $s_0(11) = 0$, $s_1(00) = 0$, $s_1(01) = s_1(10) = 1$, $s_1(11) = 2$, $o(00) = o(11) = 2$, $o(01) = o(10) = 1$. $\Diamond$

2.3.1. **(5)** Let $D$ be a link diagram with $n$ crossings. Show that

$$\langle D \rangle = \sum_{\epsilon \in \{0,1\}^n} \langle D, \epsilon \rangle ,$$

where the sum is over all smoothings $\epsilon$ of $D$.

2.3.2. **(3)** Let $D$ be a link diagram and let $\epsilon$ be a smoothing of $D$. Show that

$$\langle D_\epsilon \rangle = (-A^2 - A^{-2})^{o(\epsilon)-1}.$$

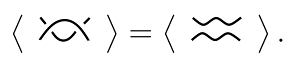
2.3.3. **(2)** Show that if $\epsilon$ and $\epsilon'$ differ by one resolution, then

$$o(\epsilon') = o(\epsilon) \pm 1.$$

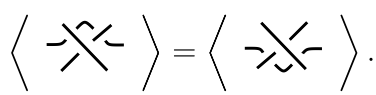
## 2.4 Invariance under Type II and Type III Moves (10 points)

Because of our discussion of invariants and Reidemeister moves at the end of Section 1.2, we should study how the bracket polynomial behaves under Reidemeister moves.

2.4.1. **(5)** Show that if link diagrams $D$ and $D'$ are related by one application of a Type II Reidemeister move, then $\langle D \rangle = \langle D' \rangle$. That is, show that



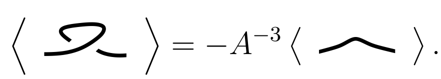
2.4.2. **(5)** Show that if link diagrams $D$ and $D'$ are related by one application of a Type III Reidemeister move, then $\langle D \rangle = \langle D' \rangle$. That is, show



## 2.5 Type I moves (3 points)

In the previous section, you showed that the bracket polynomial is invariant under Type II and Type III Reidemeister moves. If it is also invariant under Type I moves, the the bracket polynomial would be a genuine link invariant. However, this is not the case.

2.5.1. **(3)** Show that



(a) positive crossing

(b) negative crossing

Figure 2.3: Two types of crossings for an oriented diagram

## 2.6 Writhe of an oriented link (4 points)

**Definition 2.14.** Given an oriented link diagram we can define *positive crossings* and *negative crossings* by Figure 2.3. ♢

**Definition 2.15.** Let $n_+(D)$ and $n_-(D)$ be the number of positive and negative crossings, respectively, of an oriented link diagram $D$. ♢

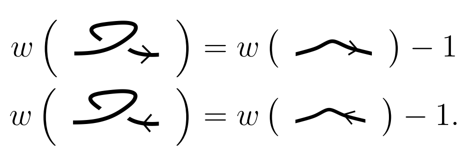
**Definition 2.16.** For an oriented link diagram $D$, the *writhe* of $D$ is $w(D) = n_+(D) - n_-(D)$. ♢

**Fact 2.17.** As the following diagrams show, if we reverse the direction of all components of a link, the crossing types (positive/negative) do not change.

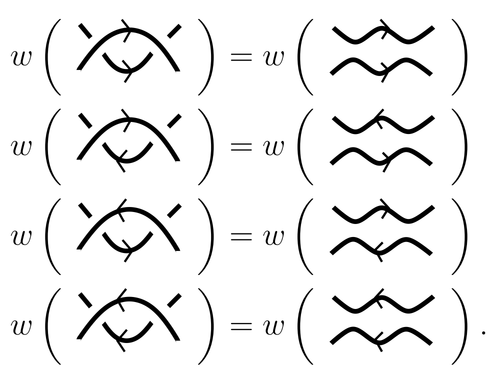
♢

**Remark 2.18.** Since a knot is a link with one component, we can define positive and negative crossings for knot diagrams without specifying an orientation on the knot. Note that this is not true for links in general. If we reverse the orientations of some (but not all) of the components of a link, then some crossing types will change. ♢

2.6.1. **(2)** Show



2.6.2. **(2)** Show



**Fact 2.19.** The writhe is also unchanged under type III moves. ♢

### 2.7 The Jones Polynomial (7 points)

Because of Problem 2.6.1, the writhe is not invariant under Type I moves.

2.7.1. **(7)** For an oriented link $L$ with diagram $D$, show that the polynomial $(-A)^{-3w(D)} \langle D \rangle$ is an invariant of the link $L$.

**Definition 2.20.** Let $L$ be an oriented link, and let $D$ be a diagram of $L$. The *Jones polynomial* of $L$, denoted $V_L(t)$, is obtained by taking the expression $(-A)^{-3w(D)} \langle D \rangle$ and setting $A = t^{-1/4}$. ♢

**Fact 2.21.** Because of Problem 2.7.1, the Jones polynomial is an invariant of oriented links. For knots, recall that the writhe does not depend on the orientation. If we retrace our arguments above, we see that the Jones polynomial is also an invariant of (unoriented) knots. ♢

**Fact 2.22.** For the trefoil (see Figure 1.1b), we have

$$V_K(t) = -t^{-4} + t^{-3} + t^{-1}.$$

You might want to verify this yourself. ♢

**Remark 2.23.** Vaughan Jones received the Fields Medal for his discovery of the Jones polynomial. It is a pretty big deal! ♢

# 3 Detecting chiral knots

## 3.1 Mirroring knots (25 points)

**Definition 3.1.** For a knot $K$, let $K^{\text{flip}}$ denote the mirrored knot. (In other words, make $K$ out of a piece of string, and hold it in front of a mirror! The knot you see in the mirror is $K^{\text{flip}}$.) ♢

**Definition 3.2.** A knot $K$ is *amphichiral* if it is equivalent to $K^{\text{flip}}$. Otherwise, it is *chiral*. ♢

Consider, for example the two diagrams in Figure 3.1. We may ask if they are the same knot.

3.1.1. **(20)** Show that for any knot $K$,

$$V_{K^{\text{flip}}}(t) = V_K(t^{-1}).$$

3.1.2. **(5)** Is the trefoil amphichiral or chiral? Justify your answer.

**Remark 3.3.** Recall that the Jones polynomial is not just a knot invariant but also an oriented link invariant. Thus, Problem 3.1.1 still holds if we replace "knot" with "oriented link." ♢

(a) diagram of $K$ (b) diagram of $K^{\text{flip}}$

Figure 3.1: Mirror images of a trefoil

# 4 Bound on crossing numbers

## 4.1 Crossing number (3 points)

**Definition 4.1.** The *crossing number* of a knot $K$ is the minimum number of crossings needed to draw the knot in a plane. It is denoted $c(K)$. ♢

4.1.1. **(3)** While the diagram given in Figure 4.1 has seven crossings, show that the crossing number of that knot is *not* 7.

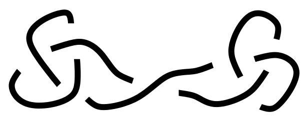


Figure 4.1

## 4.2 Reduced diagrams (5 points)

**Definition 4.2.** A knot diagram D is *un-reduced* is it has the form of Figure 4.2. (That is, there are exactly two strands in the region between $X$ and $Y$ that go from $X$ to $Y$. Furthermore, these two strands cross each other exactly once.) A knot diagram D is *reduced* if it is not un-reduced.

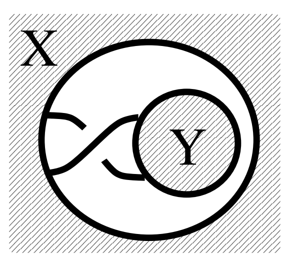



Figure 4.2

♢

4.2.1. **(5)** Show, by drawing an example, that it is possible to have two diagrams $D$ and $D'$ of the same knot $K$, such that

- $D$ is reduced and
- $D'$ has fewer crossings than $D$.

## 4.3 Knots and graphs (5 points)

Every knot diagram corresponds to a planar graph. (See, for example, Figure 4.3.)

(a) trefoil (b) corresponding graph

Figure 4.3

**Definition 4.3.** A *graph* is a set of vertices (e.g., the black dots in Figure 4.3b) that are joined together by edges (e.g., the lines between the dots in Figure 4.3b). A graph is *planar* if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. ♢

**Fact 4.4.** A planar graph divides the plane into regions, called faces. (The exterior of a graph is considered a face too.) If $V$, $E$, $F$ are the number of vertices, edges, faces (respectively) of a planar graph, then Euler's formula says that $V - E + F = 2$. ♢

**Example 4.5.** The planar graph Figure 4.3b has $V = 3$, $E = 6$, $F = 5$ (don't forget to count the exterior face!). Thus, $V - E + F = 2$, as expected. ♢

4.3.1. **(5)** Show that for a knot diagram with $n$ crossings, the corresponding graph has $n+2$ faces.

## 4.4 Coloring faces of knots (15 points)

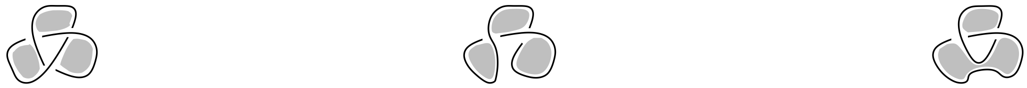
It is always possible to color the faces of a link in an alternating black-white pattern, as shown in Figure 4.4. (Recall that we are considering the exterior of the knot diagram to be a face as well, which explains why Figure 4.4b is a valid checkerboard coloring of the trefoil.)

(a) trefoil (b) trefoil

Figure 4.4: Examples of checkerboard colorings

**Fact 4.6.** By resolving a crossing, we still get a checkerboard coloring. (See, for example Figure 4.5.) ♢

Given a checkerboard coloring of a knot diagram, we can divide the crossings into two types. (See Figure 4.6.)



(a) checkerboard (b) 0-resolution (c) 1-resolution

Figure 4.5: Resolving a crossing still gives us a checkerboard coloring.

(a) 0-separating (b) 1-separating

Figure 4.6: Two coloring types at a crossing

**Definition 4.7.** The coloring in Figure 4.6a is called "0-separating" (because a 0-resolution separates the two black regions). The coloring in Figure 4.6b is called "1-separating." ♢

**Definition 4.8.** A knot diagram is *alternating* if the strand alternates between going over and going under at crossings. A knot is *alternating* if there is a alternating diagram of the knot. ♢

4.4.1. **(15)** Show that a knot diagram is alternating if and only if the diagram admits a checkerboard coloring consisting of only 0-separating crossings.

## 4.5 The span of the bracket polynomial (18 points)

Recall the definitions of $\langle D, \epsilon \rangle$ and $o(\epsilon)$ given in section Section 2.3.

**Definition 4.9.** For a Laurent polynomial $f(x)$, we define $\mathrm{hp}(f)$ to be the highest power of $x$ that appears in $f$, and we define $\mathrm{lp}(f)$ similarly to be the lowest power. We define $\mathrm{span}(f) = \mathrm{hp}(f) - \mathrm{lp}(f)$. ♢

**Definition 4.10.** Let $D$ be a link diagram. We let $\mathbf{0} = (0, 0, \ldots, 0)$ denote the smoothing with all 0-resolutions and $\mathbf{1} = (1, 1, \ldots, 1)$ denote the smoothing with all 1-resolutions. ♢

4.5.1. **(5)** Let $D_1$ and $D_2$ be knot diagrams of the same knot. Show that $\mathrm{span}\,\langle D_1 \rangle = \mathrm{span}\,\langle D_2 \rangle$.

4.5.2. **(3)** Let $D$ be a knot diagram and let $\epsilon$ be a smoothing of $D$. Show that

$$\mathrm{hp}\,\langle D, \epsilon \rangle = s_0(\epsilon) - s_1(\epsilon) + 2o(\epsilon) - 2.$$

4.5.3. **(5)** Let $D$ be a knot diagram. Show that

$$\mathrm{hp}\,\langle D, \mathbf{0} \rangle \geq \mathrm{hp}\,\langle D, \epsilon \rangle$$

for all smoothings $\epsilon$.

4.5.4. **(2)** Let $D$ be a knot diagram with $n$ crossings. Show that

$$\mathrm{hp}\,\langle D\rangle \leq n + 2o(\mathbf{0}) - 2.$$

4.5.5. **(3)** Let $D$ be a knot diagram with $n$ crossings. Show that

$$\mathrm{span}\,\langle D\rangle \leq 2n + 2(o(\mathbf{0}) + o(\mathbf{1})) - 4.$$

## 4.6 Connected link diagrams (17 points)

**Definition 4.11.** We say that a link diagram is *connected* if its corresponding graph is connected. For example, the usual diagram for the unlink ○○ is not connected. However, if the two components overlap in the diagram (as in ), then the diagram is connected. (Note that knot diagrams are always connected.) ♢

4.6.1. **(15)** Let $D$ be a connected link diagram. Show that if $D$ has $n$ crossings, then

$$o(\mathbf{0}) + o(\mathbf{1}) \leq n + 2.$$

4.6.2. **(2)** Let $D$ be a knot diagram with $n$ crossings. Show that

$$\mathrm{span}\,\langle D\rangle \leq 4n.$$

## 4.7 Reduced alternating knot diagrams (42 points)

The goal of this section is to show that alternating knots are nice.

**Definition 4.12.** A knot diagram $D$ is *reduced alternating* if it is both reduced and alternating. ♢

**Fact 4.13.** Given an alternating knot $K$, there is a diagram $D$ of $K$ that is reduced alternating. (This is not hard to show.) ♢

4.7.1. **(7)** Show that if $D$ is a reduced alternating diagram and $\epsilon$ is a smoothing with exactly one 1-resolution, then

$$o(\mathbf{0}) = o(\epsilon) + 1.$$

4.7.2. **(5)** Show that the assumption that the diagram $D$ is reduced is necessary for Problem 4.7.1. Give an example where $D$ is alternating but $o(\mathbf{0}) \neq o(\epsilon) + 1$.

4.7.3. **(30)** Let $D$ be a reduced alternating knot diagram with $n$ crossings. Show that

$$\mathrm{span}\,\langle D\rangle = 4n.$$

## 4.8 Back to the Jones polynomial (13 points)

We are almost there!

4.8.1. **(3)** Let $K$ be a knot. Show that

$$c(K) \geq \operatorname{span} V_K.$$

4.8.2. **(5)** Let $K$ be an alternating knot. Show that

$$c(K) = \operatorname{span} V_K.$$

4.8.3. **(5)** Determine the crossing number $c(K)$ of the knot $K$ depicted by the diagram $D$ in Figure 4.7. Justify your answer.

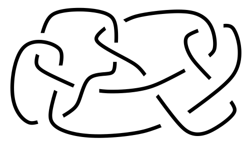


Figure 4.7: $D$, crazy knot diagram

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