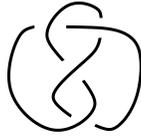


# PUMaC 2013 Power Round Solutions



## Rules, Remarks, and Reminders

You probably don't need rules for the solutions, so I'll use this space to thank people instead!

## Acknowledgments

I took the contents of the Power Round from my junior paper (JP), which I wrote during my spring 2013 term at Princeton University. I had fun writing this year's Power Round, so I would like to thank the following people for making this possible.

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3. Andy Loo (Problem Czar) and the PUMaC testers: for reviewing drafts of the Power Round and testing out the problems.
4. PUMaC competitors: for taking the Power Round!

–Alan Chang ☺

Fun fact: You can make a happy face in L<sup>A</sup>T<sub>E</sub>X with `\ddot{smile}`. This is because `\ddot{x}` gives you  $\ddot{x}$  (notation for second derivative) and `\smile` gives you ☺.

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# 1 Preliminaries

## 1.1 Definitions

Although there are more technical definitions, for this Power Round, it is enough to think of a *knot* as something made physically by attaching the two ends of a string together. Since knots exist in three dimensions, when we need to draw them on paper, we often use *knot diagrams*. Figure 1.1 contains examples of knot diagrams.

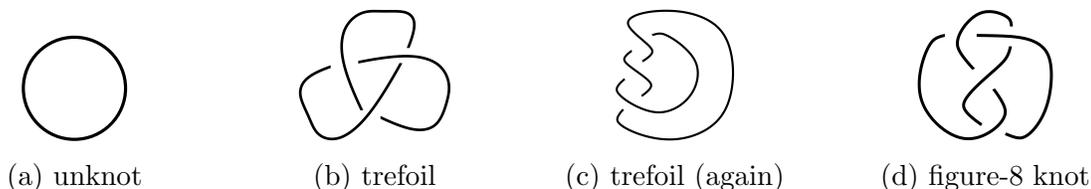


Figure 1.1: Examples of knot diagrams, with the names of the knots they represent.

As we can see from Figure 1.1b and Figure 1.1c, different knot diagrams can represent the same knot. To see that these two are really the same knot, we could make Figure 1.1b out of a piece of string and move the string around in space (without cutting it) so that it looks like Figure 1.1c.

**Tip 1.1.** Make sure you understand the distinction between “knot” and “knot diagram”! Do *not* use these terms interchangeably in your solutions. If you use one when you mean the other, the proof will be incorrect (logically) and this will lead to a significantly lower score.  $\diamond$

There are some restrictions on knot diagrams: (1) each crossing must involve exactly two segments of the string and (2) those segments must cross transversely. (See Figure 1.2.)



Figure 1.2: Examples of invalid knot diagrams

There are two ways to travel around a knot; these correspond to the *orientations* of the knot. An *oriented* knot is a knot with a specified orientation. On a knot diagram, we can indicate an orientation via an arrow. (See Figure 1.3.)

Sometimes we’ll use more than one piece of string, so we define a *link* to be a generalization of a knot: links can be made by multiple pieces of string. For each string, we attach the two ends together. (Note that we do not attach the ends of two different strings together.)

The number of *components* of a link is the number of strings used. (Observe that every knot is a link with one component.) A *link diagram* is a straightforward generalization

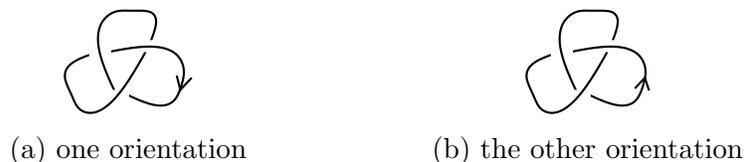


Figure 1.3: Two orientations of the trefoil

of a knot diagram, and an *oriented link* is a link where all the components have specified orientations.

Figure 1.4 contains examples of two-component links.

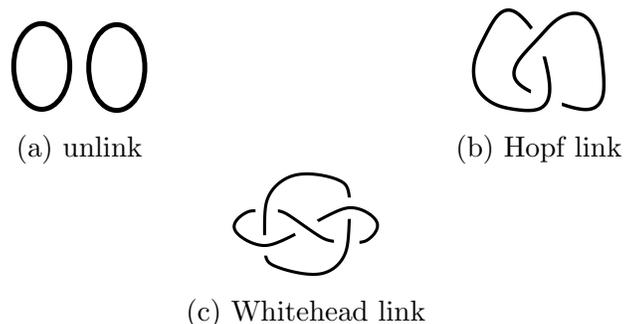


Figure 1.4: Examples of links with two components

**Tip 1.2.** Pay close attention to the problem statements in this Power Round. If a problem asks you to prove something for links, it is not enough to prove it for knots!  $\diamond$

A *knot invariant* is something (such as number, matrix, or polynomial) associated to a knot. A *link invariant* is defined similarly for links. An example of a knot invariant is the *crossing number*, which we will now define.

Suppose for a knot  $K$ , we take a diagram  $D$  of  $K$  and count the number of crossings in  $D$ . This number is *not* an invariant of  $K$  because  $K$  has many different diagrams that differ in number of crossings. For example, in Figure 1.5, we see two different diagrams of the unknot.



Figure 1.5: Two diagrams of the unknot, with different number of crossings.

Thus, we have not yet successfully defined a knot invariant. However, if we consider *all* diagrams of  $K$  and take the *minimum* number of crossings over all diagrams, then we do have an invariant of  $K$ . This is called the *crossing number* of a knot.

## 1.2 Reidemeister moves

Consider the kinds of moves in Figure 1.6, which you can perform on a knot diagram. These are called *Reidemeister moves*. We can think of Type I as adding/removing a twist, Type II as crossing/uncrossing two strands, and Type III as sliding a strand past a crossing.

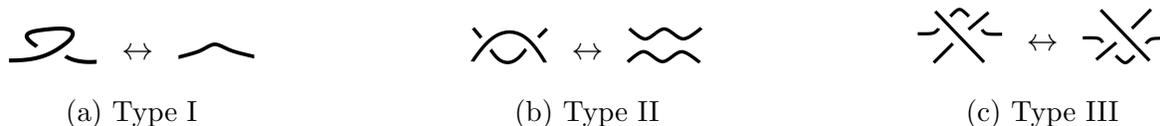


Figure 1.6: The three types of Reidemeister moves

**Fact 1.3.** Suppose we start with a knot diagram  $D_1$  and perform one of the Reidemeister moves on it so that we end up with a knot diagram  $D_2$ . The knot diagrams  $D_1$  and  $D_2$  might be different, but they will represent the same knot. (This is clear from the diagrams in Figure 1.6.)  $\diamond$

**Fact 1.4.** In 1926, Kurt Reidemeister proved that given two diagrams  $D_1$  and  $D_2$  of the same link, it is always possible to get from  $D_1$  to  $D_2$  via a finite sequence of Reidemeister moves. This is a remarkable fact!  $\diamond$

**Remark 1.5.** Suppose there is a quantity that we are trying to show is a knot invariant, but it is defined in terms of a knot diagram. There is a possibility that the quantity is different for different diagrams of the same knot, in which case our quantity would not be a knot invariant. However, if we can show the quantity is unchanged when we alter the diagram via any Reidemeister move, then we know it is an invariant, because of Fact 1.4.  $\diamond$

## 2 The Jones Polynomial

### 2.1 Resolving a crossing (4 points)

**Definition 2.1.** Suppose we start with a crossing of the form  $\times$ . The 0-resolution of this crossing is  $\succ\langle$  and the 1-resolution is  $\succsim$ . (For example, see Figure 2.1.)  $\diamond$

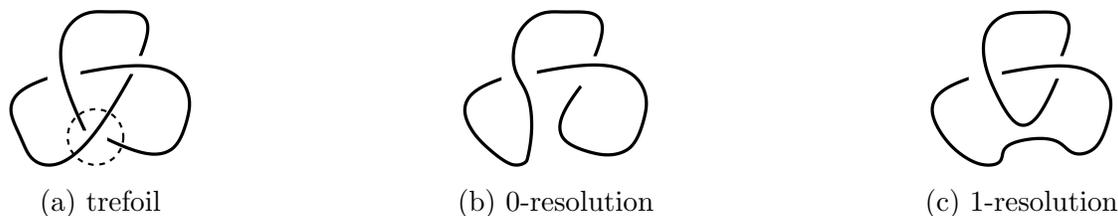


Figure 2.1: Resolving a crossing

**Remark 2.2.** A diagram may need to be rotated so that the crossing in concern appears as  $\times$ . For example, after a  $90^\circ$  rotation, we see that the 0- and 1-resolutions of  $\times$  are  $\succ$  and  $\succleftarrow$ , respectively.  $\diamond$

**Remark 2.3.** Here is one way to think of a 0-resolution: if we are traveling along a knot and reach a crossing in which we are on the upper strand, then we turn left onto the lower strand. (For a 1-resolution, we would turn right instead.)  $\diamond$

2.1.1. **Problem: (2)** Start with the diagram of the figure 8 knot given in Figure 2.2. Observe that the crossings are labeled  $A, B, C, D$ . Draw the diagram in which crossing  $B$  has been resolved with a 0-resolution and identify the resulting knot/link.

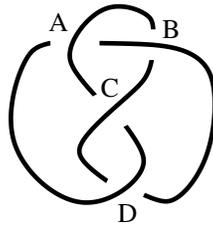
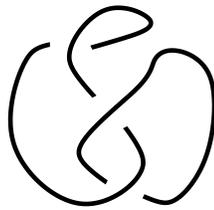


Figure 2.2

**Solution:** The diagram becomes



It is a Hopf link.

2.1.2. **Problem: (2)** Once again, start with the diagram of the figure 8 knot given in Figure 2.2. It is possible to resolve one of the crossings so that the resulting diagram is a trefoil. Which crossing can we resolve ( $A, B, C, D$ ) and how do we resolve it (0- or 1-resolution)? Simply state an answer.

**Solution:** There are four possible solutions. Any one is enough.

- Resolve crossing  $A$  or  $B$  with a 1-resolution.
- Resolve crossing  $C$  or  $D$  with a 0-resolution.

## 2.2 The Bracket Polynomial (0 points)

**Definition 2.4.** As you all know, a polynomial in  $x$  is something of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

A *Laurent polynomial* in  $x$  is like a polynomial except you can use negative powers. In other words a Laurent polynomial is something of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_m x^m,$$

where  $m$  and  $n$  are integers with  $m \leq n$ . (For example,  $x^3 + 2x + 4 + x^{-1} - 5x^{-4}$  is a Laurent polynomial.) Quite confusingly, a Laurent polynomial is not necessarily a polynomial.  $\diamond$

**Definition 2.5.** The *bracket polynomial* of a link diagram  $D$  is a Laurent polynomial in the variable  $A$  and is denoted  $\langle D \rangle$ . It is completely determined by three rules:

$$\langle \bigcirc \rangle = 1 \quad (\text{BP1})$$

$$\langle D \sqcup \bigcirc \rangle = (-A^2 - A^{-2}) \langle D \rangle \quad (\text{BP2})$$

$$\langle \times \rangle = A \langle \rangle \langle \rangle + A^{-1} \langle \asymp \rangle \quad (\text{BP3})$$

(The BP stands for “bracket polynomial.”)  $\diamond$

Let’s go through what these rules mean, one by one.

1. The first relation (BP1) states that the bracket polynomial of the knot diagram  $\bigcirc$  is the constant polynomial 1. (Note, however, that this does *not* mean that the bracket polynomial of *any* diagram depicting the unknot is 1. For example,  is also a diagram of the unknot, but this diagram turns out *not* to have bracket polynomial 1.)
2. For the second relation, the expression  $D \sqcup \bigcirc$  denotes a diagram  $D$  with an extra circle added. Furthermore, the circle does not cross the rest of the diagram. If we do have a diagram of this form, then BP2 means that we can find its bracket polynomial by starting with the bracket polynomial of the diagram with the circle removed and multiplying it by  $-A^2 - A^{-2}$ . For example, using BP2 (along with BP1), we have

$$\langle \bigcirc \bigcirc \rangle = (-A^2 - A^{-2}) \langle \bigcirc \rangle = -A^2 - A^{-2}.$$

3. In order to apply the third relation, we need to resolve crossings. Start with a diagram  $D$  and fix a crossing. If  $D_0$  and  $D_1$  are the 0- and 1-resolutions of this crossing, then BP3 states that  $\langle D \rangle = A \langle D_0 \rangle + A^{-1} \langle D_1 \rangle$ . For example,

$$\langle \text{Diagram with crossing} \rangle = A \langle \text{Diagram with no crossing} \rangle + A^{-1} \langle \text{Diagram with crossing} \rangle$$

**Example 2.6.** Let’s compute the bracket polynomial of the diagram . (It is a diagram of the Hopf link.) Applying BP3 gives us

$$\langle \text{Hopf link} \rangle = A \langle \text{Resolution 1} \rangle + A^{-1} \langle \text{Resolution 2} \rangle. \quad (2.1)$$

Using BP3 again,

$$\begin{aligned}\langle \text{Diagram 1} \rangle &= A \langle \text{Diagram 2} \rangle + A^{-1} \langle \text{Diagram 3} \rangle \\ \langle \text{Diagram 4} \rangle &= A \langle \text{Diagram 5} \rangle + A^{-1} \langle \text{Diagram 6} \rangle.\end{aligned}\tag{2.2}$$

Combining (2.1) and (2.2), we see that

$$\langle \text{Diagram 7} \rangle = A^2 \langle \text{Diagram 2} \rangle + \langle \text{Diagram 3} \rangle + \langle \text{Diagram 4} \rangle + A^{-2} \langle \text{Diagram 6} \rangle.$$

Invoking BP1 and BP2 gives us

$$\langle \text{Diagram 3} \rangle = \langle \text{Diagram 4} \rangle = 1 \quad \text{and} \quad \langle \text{Diagram 2} \rangle = \langle \text{Diagram 6} \rangle = -A^2 - A^{-2}.$$

Putting everything together gives us  $\langle \text{Diagram 7} \rangle = -A^4 - A^{-4}$ . ◇

**Fact 2.7.** For the trefoil, we have

$$\langle \text{Trefoil} \rangle = A^7 - A^3 - A^{-5}.$$

You might want to verify this yourself, to make sure you understand how to compute bracket polynomials of knot diagrams. ◇

## 2.3 Smoothings (10 points)

In Example 2.6, we decomposed the Hopf link into four diagrams. The four diagrams correspond to the four ways of resolving the two crossings of  $\text{Diagram 7}$ . Each of these diagrams is called a *smoothing*.

**Definition 2.8.** Given a link diagram  $D$ , a *smoothing* of  $D$  is a diagram in which every crossing of  $D$  has been resolved (either by a 0-resolution or a 1-resolution). Note that a smoothing has no crossings. ◇

**Definition 2.9.** Let  $\{0, 1\}^n$  denote the set of  $n$ -tuples, where each component is either 0 or 1. ◇

**Definition 2.10.** Let  $D$  be a link diagram with  $n$  crossings. Number the crossings  $1, \dots, n$ . Let  $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$ . Then  $D_\epsilon$  denotes the smoothing of  $D$  where crossing  $i$  is resolved via a  $\epsilon_i$ -resolution for  $i = 1, \dots, n$ . (Note that  $D_\epsilon$  is also a knot diagram.) ◇

**Definition 2.11.** Let  $D$  be a link diagram, and let  $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \{0, 1\}^n$  be a smoothing of  $D$ . Define

$$\begin{aligned}s_0(\epsilon) &= \text{the number of 0-resolutions in } D_\epsilon \\ s_1(\epsilon) &= \text{the number of 1-resolutions in } D_\epsilon \\ o(\epsilon) &= \text{the number of circles in } D_\epsilon.\end{aligned}$$

(We use the letter  $o$  because it looks like a circle!) Also, define

$$\langle D, \epsilon \rangle = A^{s_0(\epsilon) - s_1(\epsilon)} \langle D_\epsilon \rangle. \quad \diamond$$

**Remark 2.12.** We will omit the commas in the  $(\epsilon_1, \dots, \epsilon_n)$  notation to avoid clutter. For example,  $\epsilon = 10011$  is short for  $\epsilon = (1, 0, 0, 1, 1)$ .  $\diamond$

**Example 2.13.** If  $D = \mathcal{O}$ , and the top crossing is labeled 1 (so the bottom crossing is labeled 2), then

$$D_{00} = \mathcal{O}, \quad D_{01} = \mathcal{O}, \quad D_{10} = \mathcal{O}, \quad D_{11} = \mathcal{O}. \quad (2.3)$$

Also,  $s_0(00) = 2$ ,  $s_0(01) = s_0(10) = 1$ ,  $s_0(11) = 0$ ,  $s_1(00) = 0$ ,  $s_1(01) = s_1(10) = 1$ ,  $s_1(11) = 2$ ,  $o(00) = o(11) = 2$ ,  $o(01) = o(10) = 1$ .  $\diamond$

2.3.1. **Problem: (5)** Let  $D$  be a link diagram with  $n$  crossings. Show that

$$\langle D \rangle = \sum_{\epsilon \in \{0,1\}^n} \langle D, \epsilon \rangle,$$

where the sum is over all smoothings  $\epsilon$  of  $D$ .

**Solution:** Start with  $\langle D \rangle$  and repeatedly apply **BP3** until none of the diagrams have any more crossings. The result is a sum over all the smoothings  $\epsilon$  of  $D$ .

The coefficient  $A$  of  $\langle \rangle \langle \rangle$  and the coefficient of  $A^{-1}$  of  $\langle \asymp \rangle$  in **BP3** means that the coefficient of  $\langle D_\epsilon \rangle$  will be  $A^{s_0(\epsilon) - s_1(\epsilon)}$ .

2.3.2. **Problem: (3)** Let  $D$  be a link diagram and let  $\epsilon$  be a smoothing of  $D$ . Show that

$$\langle D_\epsilon \rangle = (-A^2 - A^{-2})^{o(\epsilon) - 1}.$$

**Solution:** We know  $D_\epsilon$  has  $o(\epsilon)$  circles. Apply **BP2** over and over again (a total of  $o(\epsilon) - 1$  times), then apply **BP1** once to get the the polynomial  $(-A^2 - A^{-2})^{o(\epsilon) - 1}$ .

2.3.3. **Problem: (2)** Show that if  $\epsilon$  and  $\epsilon'$  differ by one resolution, then

$$o(\epsilon') = o(\epsilon) \pm 1.$$

**Solution:** Changing one resolution either merges two circles in  $D_\epsilon$  or separates one circle into two.

## 2.4 Invariance under Type II and Type III Moves (10 points)

Because of our discussion of invariants and Reidemeister moves at the end of Section 1.2, we should study how the bracket polynomial behaves under Reidemeister moves.

2.4.1. **Problem: (5)** Show that if link diagrams  $D$  and  $D'$  are related by one application of a Type II Reidemeister move, then  $\langle D \rangle = \langle D' \rangle$ . That is, show that

$$\langle \text{Type II move} \rangle = \langle \text{Type II move} \rangle.$$

**Solution:** Start with the diagram  $D = \text{Type II move}$ . Label the two crossings so that 1 is on the left and 2 is on the right. The four ways of resolving these crossings are given by

$$D_{00} = \text{Type II move}, \quad D_{01} = \text{Type II move}, \quad D_{10} = \text{Type II move}, \quad D_{11} = \text{Type II move}.$$

Following Example 2.6 or using Problem 2.3.1, we have

$$\langle \text{Type II move} \rangle = A^2 \langle \text{Type II move} \rangle + \langle \text{Type II move} \rangle + \langle \text{Type II move} \rangle + A^{-2} \langle \text{Type II move} \rangle. \quad (2.A)$$

Using BP2, we get  $\langle \text{Type II move} \rangle = (-A^2 - A^{-2}) \langle \text{Type II move} \rangle$ , so three of the four terms on the right hand side of (2.A) cancel out:

$$A^2 \langle \text{Type II move} \rangle + \langle \text{Type II move} \rangle + A^{-2} \langle \text{Type II move} \rangle = A^2 \langle \text{Type II move} \rangle + ((-A^2 - A^{-2}) \langle \text{Type II move} \rangle) + A^{-2} \langle \text{Type II move} \rangle = 0.$$

We are left with  $\langle \text{Type II move} \rangle = \langle \text{Type II move} \rangle$ . This proves that the bracket polynomial is unchanged under a Type II move.

2.4.2. **Problem: (5)** Show that if link diagrams  $D$  and  $D'$  are related by one application of a Type III Reidemeister move, then  $\langle D \rangle = \langle D' \rangle$ . That is, show

$$\langle \text{Type III move} \rangle = \langle \text{Type III move} \rangle.$$

**Solution:** We could start with the diagram  $\text{Type III move}$  and relate it to the 8 diagrams that correspond to the 8 possible ways of resolving the three crossings. Then we could do the same thing with  $\text{Type III move}$ . We would indeed see that both diagrams give the same result.

However, we can make a cleaner argument by noting what happens if we only resolve the crossing between the two upper strands:

$$\begin{aligned} \langle \text{Type III move} \rangle &= A \langle \text{Type III move} \rangle + A^{-1} \langle \text{Type III move} \rangle \\ \langle \text{Type III move} \rangle &= A \langle \text{Type III move} \rangle + A^{-1} \langle \text{Type III move} \rangle \end{aligned}$$

To complete the proof, it suffices to show that

$$\left\langle \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right\rangle = \left\langle \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \right\rangle \quad \text{and} \quad \left\langle \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right\rangle = \left\langle \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \right\rangle$$

The equality on the left holds by two applications of Problem 2.4.1. The equality on the right holds because we can slide the bottom strand.

## 2.5 Type I moves (3 points)

In the previous section, you showed that the bracket polynomial is invariant under Type II and Type III Reidemeister moves. If it is also invariant under Type I moves, the the bracket polynomial would be a genuine link invariant. However, this is not the case.

2.5.1. **Problem: (3)** Show that

$$\left\langle \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right\rangle = -A^{-3} \left\langle \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right\rangle.$$

**Solution:** We have

$$\begin{aligned} \left\langle \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right\rangle &= A \left\langle \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \curvearrowleft \\ \curvearrowleft \end{array} \right\rangle \\ &= A \left\langle \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right\rangle + A^{-1}(-A^2 - A^{-2}) \left\langle \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right\rangle \\ &= -A^{-3} \left\langle \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right\rangle. \end{aligned}$$

## 2.6 Writhe of an oriented link (4 points)

**Definition 2.14.** Given an oriented link diagram we can define *positive crossings* and *negative crossings* by Figure 2.3.  $\diamond$

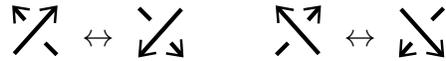


Figure 2.3: Two types of crossings for an oriented diagram

**Definition 2.15.** Let  $n_+(D)$  and  $n_-(D)$  be the number of positive and negative crossings, respectively, of an oriented link diagram  $D$ .  $\diamond$

**Definition 2.16.** For an oriented link diagram  $D$ , the *writhe* of  $D$  is  $w(D) = n_+(D) - n_-(D)$ .  $\diamond$

**Fact 2.17.** As the following diagrams show, if we reverse the direction of all components of a link, the crossing types (positive/negative) do not change. ◇



**Remark 2.18.** Since a knot is a link with one component, we can define positive and negative crossings for knot diagrams without specifying an orientation on the knot. Note that this is not true for links in general. If we reverse the orientations of some (but not all) of the components of a link, then some crossing types will change. ◇

2.6.1. **Problem: (2)** Show

$$w \left( \begin{array}{c} \curvearrowright \\ \rightarrow \end{array} \right) = w \left( \begin{array}{c} \curvearrowright \\ \rightarrow \end{array} \right) - 1$$

$$w \left( \begin{array}{c} \curvearrowleft \\ \leftarrow \end{array} \right) = w \left( \begin{array}{c} \curvearrowleft \\ \leftarrow \end{array} \right) - 1.$$

**Solution:** On the left-hand side, there is a negative crossing, which contributes  $-1$  to the writhe. On the right-hand side, the negative crossing is removed, while the rest of the link diagram remains unchanged.

2.6.2. **Problem: (2)** Show

$$w \left( \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) = w \left( \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \\ \curvearrowleft \end{array} \right)$$

$$w \left( \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \right) = w \left( \begin{array}{c} \curvearrowleft \\ \curvearrowleft \\ \curvearrowright \\ \curvearrowright \end{array} \right)$$

$$w \left( \begin{array}{c} \curvearrowleft \\ \curvearrowleft \end{array} \right) = w \left( \begin{array}{c} \curvearrowleft \\ \curvearrowleft \\ \curvearrowright \\ \curvearrowright \end{array} \right)$$

$$w \left( \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) = w \left( \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \\ \curvearrowleft \end{array} \right).$$

**Solution:** Take a look at the first line. The diagram on the left-hand-side has one positive crossing and one negative crossing, so this part of the knot diagram contributes nothing to the writhe. Clearly on the right-hand side, there is no contribution to the writhe either. The same observation holds for the other three lines.

**Fact 2.19.** The writhe is also unchanged under type III moves. ◇

## 2.7 The Jones Polynomial (7 points)

Because of Problem 2.6.1, the writhe is not invariant under Type I moves.

2.7.1. **Problem:** (7) For an oriented link  $L$  with diagram  $D$ , show that the polynomial  $(-A)^{-3w(D)} \langle D \rangle$  is an invariant of the link  $L$ .

**Solution:** Because of Remark 1.5, it suffices to show that  $(-A)^{-3w(D)} \langle D \rangle$  is unchanged under the three types of Reidemeister moves.

Because both the writhe and the bracket polynomial are invariant under Type II and Type III moves, we only need to check that the quantity  $(-A)^{-3w(D)} \langle D \rangle$  is unchanged under Type I moves. This is immediate from (2.5.1) and (2.6.1):

$$\begin{aligned} (-A)^{-3w(\mathcal{R})} \langle \mathcal{R} \rangle &= (-A)^{-3w(\frown)+3} \cdot (-A)^{-3} \langle \frown \rangle \\ &= (-A)^{-3w(\frown)} \langle \frown \rangle. \end{aligned}$$

**Definition 2.20.** Let  $L$  be an oriented link, and let  $D$  be a diagram of  $L$ . The *Jones polynomial* of  $L$ , denoted  $V_L(t)$ , is obtained by taking the expression  $(-A)^{-3w(D)} \langle D \rangle$  and setting  $A = t^{-1/4}$ .  $\diamond$

**Fact 2.21.** Because of Problem 2.7.1, the Jones polynomial is an invariant of oriented links. For knots, recall that the writhe does not depend on the orientation. If we retrace our arguments above, we see that the Jones polynomial is also an invariant of (unoriented) knots.  $\diamond$

**Fact 2.22.** For the trefoil (see Figure 1.1b), we have

$$V_K(t) = -t^{-4} + t^{-3} + t^{-1}.$$

You might want to verify this yourself.  $\diamond$

**Remark 2.23.** Vaughan Jones received the Fields Medal for his discovery of the Jones polynomial. It is a pretty big deal!  $\diamond$

## 3 Detecting chiral knots

### 3.1 Mirroring knots (25 points)

**Definition 3.1.** For a knot  $K$ , let  $K^{\text{flip}}$  denote the mirrored knot. (In other words, make  $K$  out of a piece of string, and hold it in front of a mirror! The knot you see in the mirror is  $K^{\text{flip}}$ .)  $\diamond$

**Definition 3.2.** A knot  $K$  is *amphichiral* if it is equivalent to  $K^{\text{flip}}$ . Otherwise, it is *chiral*.  $\diamond$

Consider, for example the two diagrams in Figure 3.1. We may ask if they are the same knot.



Figure 3.1: Mirror images of a trefoil

3.1.1. **Problem: (20)** Show that for any knot  $K$ ,

$$V_{K^{\text{flip}}}(t) = V_K(t^{-1}).$$

**Solution:** Suppose we start with a smoothing  $D_\epsilon$  of  $D$ . What do we get when we mirror this to  $(D_\epsilon)^{\text{flip}}$ ? The answer is a smoothing of  $D^{\text{flip}}$ . See the example in Figure 3.A.

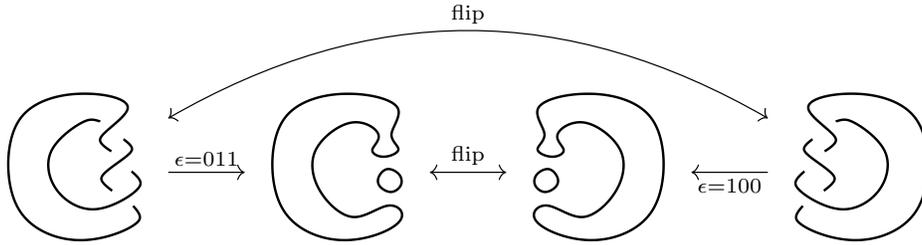


Figure 3.A: Relations between mirroring and smoothing. (In the labeling, the three crossings are numbered from top to bottom.)

The example makes a general pattern clear. Every smoothing  $\epsilon$  of  $D$  corresponds to the dual smoothing  $\hat{\epsilon}$  of  $D^{\text{flip}}$ . The dual smoothing  $\hat{\epsilon}$  is obtained by reversing every resolution in  $\epsilon$ . (That is, interchange 0s and 1s.) For example, if  $\epsilon = 011$ , then  $\hat{\epsilon} = 100$ . We can write the relation as

$$(D^{\text{flip}})_{\hat{\epsilon}} = D_\epsilon. \tag{3.A}$$

Observe that  $s_0(\epsilon) = s_1(\hat{\epsilon})$  and  $s_1(\epsilon) = s_0(\hat{\epsilon})$ . Using (3.A), it follows that

$$\langle D^{\text{flip}} \rangle(A) = \sum_{\epsilon \in \{0,1\}^n} A^{s_0(\epsilon) - s_1(\epsilon)} \langle (D^{\text{flip}})_\epsilon \rangle(A) = \sum_{\epsilon \in \{0,1\}^n} A^{-(s_0(\hat{\epsilon}) - s_1(\hat{\epsilon}))} \langle D_{\hat{\epsilon}} \rangle(A).$$

Recall that the bracket polynomial of a smoothing  $\langle D_\epsilon \rangle(A)$  is some power of  $(-A^2 - A^{-2})$ . Hence,  $\langle D_\epsilon \rangle(A) = \langle D_\epsilon \rangle(A^{-1})$ , giving us

$$\langle D^{\text{flip}} \rangle(A) = \sum_{\epsilon \in \{0,1\}^n} (A^{-1})^{s_0(\hat{\epsilon}) - s_1(\hat{\epsilon})} \langle D_{\hat{\epsilon}} \rangle(A^{-1}) = \langle D \rangle(A^{-1}).$$

Also,  $w(D) = -w(D^{\text{flip}})$ , since positive crossings in  $D$  become negative crossings in  $D^{\text{flip}}$  and vice versa. We can translate the results we have just obtained to a statement about the Jones polynomial.

3.1.2. **Problem: (5)** Is the trefoil amphichiral or chiral? Justify your answer.

**Solution:** The trefoil is chiral. Let  $K$  be the knot represented by Figure 3.1a. By Fact 2.22, we have  $V_K(t) = -t^{-4} + t^{-3} + t^{-1}$ .

It follows that the Jones polynomial of the mirrored knot is  $V_{K^{\text{flip}}}(t) = -t^4 + t^3 + t$ . Since the two polynomials are not the same, we can conclude that the trefoil and its mirror are distinct. Thus, the trefoil is chiral.

(Both knots are usually referred to as the “trefoil.” If we want to make it clear which one we are talking about, we call Figure 3.1a and Figure 3.1b the “left-handed” and “right-handed” trefoils, respectively.)

**Remark 3.3.** Recall that the Jones polynomial is not just a knot invariant but also an oriented link invariant. Thus, Problem 3.1.1 still holds if we replace “knot” with “oriented link.”  $\diamond$

## 4 Bound on crossing numbers

### 4.1 Crossing number (3 points)

**Definition 4.1.** The *crossing number* of a knot  $K$  is the minimum number of crossings needed to draw the knot in a plane. It is denoted  $c(K)$ .  $\diamond$

4.1.1. **Problem: (3)** While the diagram given in Figure 4.1 has seven crossings, show that the crossing number of that knot is *not* 7.



Figure 4.1

**Solution:** We can flip one of the two halves over to get a diagram with six crossings. (For example, flipping the right half over leaves us with Figure 4.A.) This shows that the crossing number of the knot is at most 6.

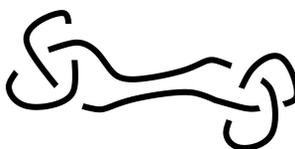


Figure 4.A

## 4.2 Reduced diagrams (5 points)

**Definition 4.2.** A knot diagram  $D$  is *un-reduced* if it has the form of Figure 4.2. (That is, there are exactly two strands in the region between  $X$  and  $Y$  that go from  $X$  to  $Y$ . Furthermore, these two strands cross each other exactly once.) A knot diagram  $D$  is *reduced* if it is not un-reduced.

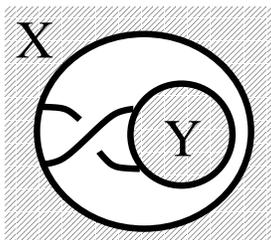


Figure 4.2

◇

4.2.1. **Problem: (5)** Show, by drawing an example, that it is possible to have two diagrams  $D$  and  $D'$  of the same knot  $K$ , such that

- $D$  is reduced and
- $D'$  has fewer crossings than  $D$ .

**Solution:** Here is one possible answer. Let  $D'$  be the unknot in its usual form:  $\bigcirc$ . Let  $D$  be the unknot as depicted in Figure 4.B.



Figure 4.B: Reduced diagram of the unknot with three crossings

The diagram  $D$  is reduced and contains three crossings, whereas  $D'$  contains no crossings.

## 4.3 Knots and graphs (5 points)

Every knot diagram corresponds to a planar graph. (See, for example, Figure 4.3.)

**Definition 4.3.** A *graph* is a set of vertices (e.g., the black dots in Figure 4.3b) that are joined together by edges (e.g., the lines between the dots in Figure 4.3b). A graph is *planar* if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. ◇

**Fact 4.4.** A planar graph divides the plane into regions, called faces. (The exterior of a graph is considered a face too.) If  $V$ ,  $E$ ,  $F$  are the number of vertices, edges, faces (respectively) of a planar graph, then Euler's formula says that  $V - E + F = 2$ . ◇

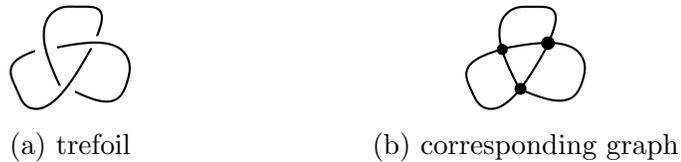


Figure 4.3

**Example 4.5.** The planar graph Figure 4.3b has  $V = 3$ ,  $E = 6$ ,  $F = 5$  (don't forget to count the exterior face!). Thus,  $V - E + F = 2$ , as expected.  $\diamond$

4.3.1. **Problem: (5)** Show that for a knot diagram with  $n$  crossings, the corresponding graph has  $n + 2$  faces.

**Solution:** Apply Euler's formula:  $V - E + F = 2$ . Because the knot diagram has  $n$  crossings, we know  $V = n$ . Because each crossing involves two strands, each vertex of the graph has four edges coming out of it. Using this fact, we know that the set of ordered pairs

$$\{(e, v) : e \text{ is an edge and } v \text{ is an endpoint of } e\}$$

has  $E \cdot 2 = V \cdot 4$  elements. Thus,  $E = 2n$  and we have  $F = n + 2$ .

#### 4.4 Coloring faces of knots (15 points)

It is always possible to color the faces of a link in an alternating black-white pattern, as shown in Figure 4.4. (Recall that we are considering the exterior of the knot diagram to be a face as well, which explains why Figure 4.4b is a valid checkerboard coloring of the trefoil.)



Figure 4.4: Examples of checkerboard colorings

**Fact 4.6.** By resolving a crossing, we still get a checkerboard coloring. (See, for example Figure 4.5.)  $\diamond$

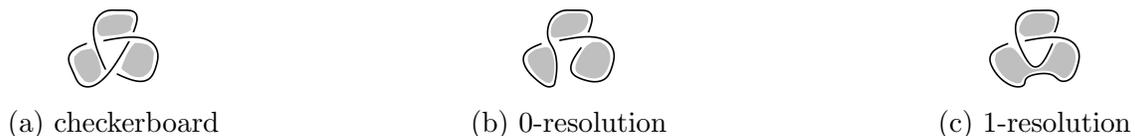


Figure 4.5: Resolving a crossing still gives us a checkerboard coloring.

Given a checkerboard coloring of a knot diagram, we can divide the crossings into two types. (See Figure 4.6.)

**Definition 4.7.** The coloring in Figure 4.6a is called “0-separating” (because a 0-resolution separates the two black regions). The coloring in Figure 4.6b is called “1-separating.”  $\diamond$



Figure 4.6: Two coloring types at a crossing

**Definition 4.8.** A knot diagram is *alternating* if the strand alternates between going over and going under at crossings. A knot is *alternating* if there is a alternating diagram of the knot.  $\diamond$

4.4.1. **Problem: (15)** Show that a knot diagram is alternating if and only if the diagram admits a checkerboard coloring consisting of only 0-separating crossings.

**Solution:** To prove the reverse direction, consider the portion of a knot diagram given in Figure 4.Ca. Suppose we start at point  $A$  and move rightwards along the horizontal strand. We will reach the crossing at the center of the diagram. (Observe that this is a 0-separating crossing.) In this crossing, the horizontal strand we are traveling along goes underneath the vertical strand.

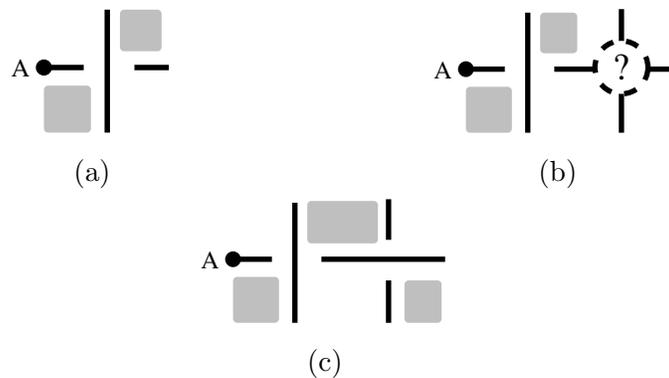


Figure 4.C

We keep moving rightwards after we pass the crossing. Eventually we will reach another crossing. (See Figure 4.Cb.) There is only one way to make this crossing 0-separating. The horizontal strand must go over the vertical strand. (See Figure 4.Cc.)

The general pattern is clear: if we want this knot diagram to have only 0-separating crossings, then the strand must alternate over-under-over-under. This completes the proof of the reverse direction.

To prove the forward direction, suppose we have an alternating knot diagram. Start at a crossing  $C$ . Color the diagram in a checkerboard pattern so that  $C$  has a 0-separating coloring. Now, consider traveling around the knot starting at  $C$ . Every time we reach a crossing, the shaded region alternates between left and right and our position at a crossing alternates between over and under. Thus, all the crossings are 0-separating.

## 4.5 The span of the bracket polynomial (18 points)

Recall the definitions of  $\langle D, \epsilon \rangle$  and  $o(\epsilon)$  given in section Section 2.3.

**Definition 4.9.** For a Laurent polynomial  $f(x)$ , we define  $\text{hp}(f)$  to be the highest power of  $x$  that appears in  $f$ , and we define  $\text{lp}(f)$  similarly to be the lowest power. We define  $\text{span}(f) = \text{hp}(f) - \text{lp}(f)$ .  $\diamond$

**Definition 4.10.** Let  $D$  be a link diagram. We let  $\mathbf{0} = (0, 0, \dots, 0)$  denote the smoothing with all 0-resolutions and  $\mathbf{1} = (1, 1, \dots, 1)$  denote the smoothing with all 1-resolutions.  $\diamond$

4.5.1. **Problem: (5)** Let  $D_1$  and  $D_2$  be knot diagrams of the same knot. Show that  $\text{span} \langle D_1 \rangle = \text{span} \langle D_2 \rangle$ .

**Solution:** There are two ways to approach this.

First solution: we only need to show that  $\text{span} \langle D \rangle$  is unchanged under the Reidemeister moves. Because  $\langle D \rangle$  is itself unchanged under type II and III moves, we only need to check type I moves. Recall that a type I move multiplies  $\langle D \rangle$  by  $(-A)^{-3}$ . This does not change the span, so we are done.

Second solution: Suppose  $D_1$  and  $D_2$  are diagrams of the knot  $K$ . From the definition of the Jones polynomial, we see that  $\text{span} \langle D_1 \rangle = 4 \text{span} V_K = \text{span} \langle D_2 \rangle$ .

4.5.2. **Problem: (3)** Let  $D$  be a knot diagram and let  $\epsilon$  be a smoothing of  $D$ . Show that

$$\text{hp} \langle D, \epsilon \rangle = s_0(\epsilon) - s_1(\epsilon) + 2o(\epsilon) - 2.$$

**Solution:**

$$\text{hp} \langle D, \epsilon \rangle = \text{hp} A^{s_0(\epsilon) - s_1(\epsilon)} (-A^2 - A^{-2})^{o(\epsilon) - 1} = s_0(\epsilon) - s_1(\epsilon) + 2o(\epsilon) - 2$$

4.5.3. **Problem: (5)** Let  $D$  be a knot diagram. Show that

$$\text{hp} \langle D, \mathbf{0} \rangle \geq \text{hp} \langle D, \epsilon \rangle$$

for all smoothings  $\epsilon$ .

**Solution:** Let  $D$  be a knot diagram and let  $\epsilon$  be a smoothing of  $D$ . Suppose we change a 0-resolution of  $\epsilon$  to a 1-resolution to get a new smoothing  $\epsilon'$ .

As we go from  $\epsilon$  to  $\epsilon'$ , the quantity  $s_0 - s_1$  decreases by 2. Since  $o(\epsilon') = o(\epsilon) \pm 1$  by Problem 2.3.3, the quantity  $2o$  can increase by 2 or decrease by 2. In either case, we have  $2o(\epsilon) \geq 2o(\epsilon') - 2$ , so

$$s_0(\epsilon) - s_1(\epsilon) + 2o(\epsilon) - 2 \geq s_0(\epsilon') - s_1(\epsilon') + 2o(\epsilon') - 2.$$

By Problem 4.5.2, it follows that  $\text{hp} \langle D, \epsilon \rangle \geq \text{hp} \langle D, \epsilon' \rangle$ .

Thus, suppose we are given a smoothing  $\tilde{\epsilon}$ . If we start at  $\mathbf{0}$  and changing the 0-resolutions to 1-resolutions one by one to get to  $\epsilon$ , the value  $\text{hp} \langle D, \epsilon \rangle$  does not increase in any of the steps.

4.5.4. **Problem: (2)** Let  $D$  be a knot diagram with  $n$  crossings. Show that

$$\text{hp} \langle D \rangle \leq n + 2o(\mathbf{0}) - 2.$$

**Solution:**  $\text{hp} \langle D \rangle \leq \max_{\epsilon} \text{hp} \langle D, \epsilon \rangle = \text{hp} \langle D, \mathbf{0} \rangle = n + 2o(\mathbf{0}) - 2$ .

4.5.5. **Problem: (3)** Let  $D$  be a knot diagram with  $n$  crossings. Show that

$$\text{span} \langle D \rangle \leq 2n + 2(o(\mathbf{0}) + o(\mathbf{1})) - 4.$$

**Solution:** Analogous to Problem 4.5.4, we have that

$$\text{lp} \langle D \rangle \geq \text{lp} \langle D, \mathbf{1} \rangle = -n - 2o(\mathbf{1}) + 2.$$

Combine this with Problem 4.5.4 to get the desired inequality.

## 4.6 Connected link diagrams (17 points)

**Definition 4.11.** We say that a link diagram is *connected* if its corresponding graph is connected. For example, the usual diagram for the unlink  $\bigcirc\bigcirc$  is not connected. However, if the two components overlap in the diagram (as in  $\bigcirc\bigcirc$ ), then the diagram is connected. (Note that knot diagrams are always connected.)  $\diamond$

4.6.1. **Problem: (15)** Let  $D$  be a connected link diagram. Show that if  $D$  has  $n$  crossings, then

$$o(\mathbf{0}) + o(\mathbf{1}) \leq n + 2.$$

**Solution:** We proceed by induction on  $n$ . For the base case ( $n = 0$ ), we have the usual diagram for the unknot ( $\bigcirc$ ). Since there are no crossings to resolve, we have  $o(D_{\mathbf{0}}) = o(D_{\mathbf{1}}) = 1$ , which completes the base case.<sup>1</sup>

---

<sup>1</sup>For this proof, we write  $o(D_{\mathbf{0}})$  instead of  $o(\mathbf{0})$  to emphasize that  $D$  is the knot we are taking the  $\mathbf{0}$ -smoothing of.

For the inductive step, suppose that the statement is true for all connected link diagrams with  $n$  crossings. Take a connected link diagram  $D$  with  $n + 1$  crossings.

Pick one of the crossings of  $D$  (call it  $x$ ). We can resolve  $x$  via a 0- or a 1-resolution; observe that at least one of the two resulting diagrams is connected. Suppose, without loss of generality, that applying the 0-resolution on  $x$  leaves us with a connected link diagram. Let this diagram be  $E$ . (See Figure 4.D.)

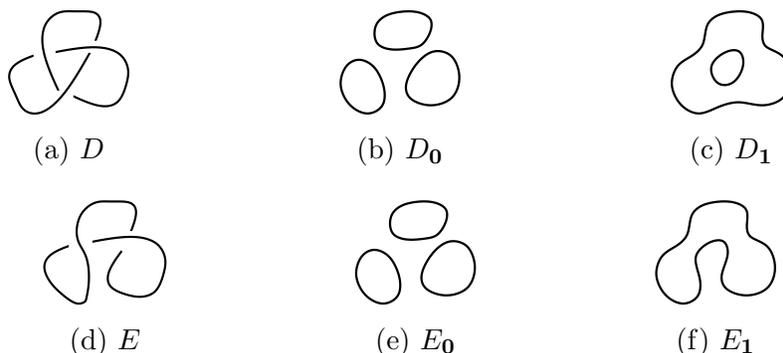


Figure 4.D: Various smoothings of  $D$  and  $E$  ( $x$  is the crossing on the bottom of  $D$ )

Note that  $E_0$  and  $E_1$  are also smoothings of  $D$ . In fact:

- $E_0 = D_0$ .
- $E_1$  and  $D_1$  differ at exactly one crossing (namely,  $x$ ).

Thus,  $o(E_0) = o(D_0)$  and  $o(E_1) = o(D_1) \pm 1$ . Furthermore,  $E$  is a connected link diagram with  $n$  crossings, so by the inductive hypothesis,  $o(E_0) + o(E_1) \leq n + 2$ . Putting the results together gives us  $o(D_0) + o(D_1) \leq n + 3$ , which completes the induction.

4.6.2. **Problem: (2)** Let  $D$  be a knot diagram with  $n$  crossings. Show that

$$\text{span} \langle D \rangle \leq 4n.$$

**Solution:** Combine Problem 4.5.5 and Problem 4.6.1.

## 4.7 Reduced alternating knot diagrams (42 points)

The goal of this section is to show that alternating knots are nice.

**Definition 4.12.** A knot diagram  $D$  is *reduced alternating* if it is both reduced and alternating.  $\diamond$

**Fact 4.13.** Given an alternating knot  $K$ , there is a diagram  $D$  of  $K$  that is reduced alternating. (This is not hard to show.)  $\diamond$

4.7.1. **Problem: (7)** Show that if  $D$  is a reduced alternating diagram and  $\epsilon$  is a smoothing with exactly one 1-resolution, then

$$o(\mathbf{0}) = o(\epsilon) + 1.$$

**Solution:** Because the diagram is alternating, we can give the diagram a checkerboard coloring so that all the crossings are 0-separating. See for example, Figure 4.Ea.

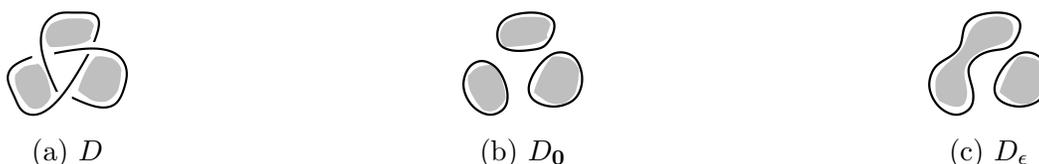


Figure 4.E

Then when we look at  $D_0$ , all the circles bound the shaded faces in the coloring. (See Figure 4.Eb.)

When we change one of the 0-resolutions to a 1-resolution, we add a “bridge” across a white region. (See Figure 4.Ec.) Because the diagram  $D$  is reduced, the bridge connects two *separate* black regions. Thus, the number of circles decreases by one, so  $o(\mathbf{0}) = o(\epsilon) + 1$ .

4.7.2. **Problem: (5)** Show that the assumption that the diagram  $D$  is reduced is necessary for Problem 4.7.1. Give an example where  $D$  is alternating but  $o(\mathbf{0}) \neq o(\epsilon) + 1$ .

**Solution:** Here’s one possible answer.

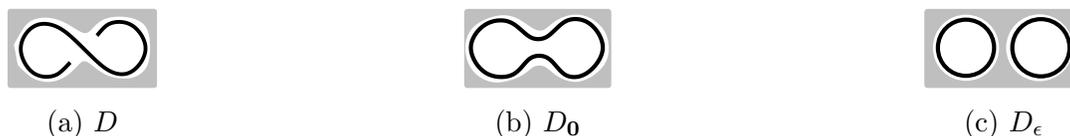


Figure 4.F

4.7.3. **Problem: (30)** Let  $D$  be a reduced alternating knot diagram with  $n$  crossings. Show that

$$\text{span} \langle D \rangle = 4n.$$

**Solution:** The key is to observe that most of the problems in the previous section have a nicer analogue for reduced alternating knot diagrams.

*Lemma 1* (analogue of Problem 4.5.3). If  $D$  is a reduced alternating diagram and  $\epsilon$  is a smoothing with exactly one 1-resolution, then  $\text{hp} \langle D, \mathbf{0} \rangle > \text{hp} \langle D, \epsilon \rangle$ .

*Proof.* Since,  $o(\mathbf{0}) = o(\epsilon) + 1$ , we can modify the proof of Problem 4.5.3 to get a strict inequality.  $\square$

*Lemma 2* (analogue of Problem 4.5.4). If  $D$  is a reduced alternating diagram, then  $\text{hp} \langle D \rangle = n + 2o(\mathbf{0}) - 2$ .

*Proof.* We have  $\langle D \rangle = \langle D, \mathbf{0} \rangle + \sum_{\epsilon \neq \mathbf{0}} \langle D, \epsilon \rangle$ . By Problem 4.5.3 and Lemma 1, we know that the highest power in  $\sum_{\epsilon \neq \mathbf{0}} \langle D, \epsilon \rangle$  is strictly lower than the highest power in  $\langle D, \mathbf{0} \rangle$ . Thus,  $\text{hp} \langle D \rangle = \text{hp} \langle D, \mathbf{0} \rangle = n + 2o(\mathbf{0}) - 2$ .  $\square$

*Lemma 3* (analogue of Problem 4.5.5). If  $D$  is a reduced alternating diagram with  $n$  crossings, then

$$\text{span} \langle D \rangle = 2n + 2(o(\mathbf{0}) + o(\mathbf{1})) - 4.$$

*Proof.* Analogous to Lemma 2, we have

$$\text{lp} \langle D \rangle = \text{lp} \langle D, \mathbf{1} \rangle = -n - 2o(\mathbf{1}) + 2.$$

Using this result together with Lemma 2 gives us the lemma.  $\square$

*Lemma 4* (analogue of Problem 4.6.1). Let  $D$  be an alternating knot. Then

$$o(\mathbf{0}) + o(\mathbf{1}) = n + 2.$$

*Proof.* Give  $D$  a checkerboard coloring with only 0-separating crossings. The key observation is that  $o(\mathbf{0})$  counts the black faces and  $o(\mathbf{1})$  counts the white faces. Thus,  $o(\mathbf{0}) + o(\mathbf{1})$  equals the total number of faces. By Problem 4.3.1, we are done.  $\square$

The equality  $\text{span} \langle D \rangle = 4n$  follows by combining Lemma 3 and Lemma 4.

## 4.8 Back to the Jones polynomial (13 points)

We are almost there!

4.8.1. **Problem: (3)** Let  $K$  be a knot. Show that

$$c(K) \geq \text{span} V_K.$$

**Solution:** By definition of  $c(K)$ , there exists a knot diagram  $D$  of  $K$  with exactly  $c(K)$  crossings. Then Problem 4.6.2 gives us  $\text{span} \langle D \rangle \leq 4c(K)$ .

4.8.2. **Problem: (5)** Let  $K$  be an alternating knot. Show that

$$c(K) = \text{span} V_K.$$

**Solution:** Note: this proof is short, but it's a little tricky. All the steps must be in the proper order!

Let  $K$  be an alternating knot. There exists a reduced alternating knot diagram  $D$  of  $K$ . Let  $n$  be the number of crossings of  $D$ . By the definition of  $c(K)$ , we know  $c(K) \leq n$ .

By Problem 4.7.3, we know  $\text{span} \langle D \rangle = 4n$ , which gives

$$\text{span } V_K = n.$$

By Problem 4.8.1, we have

$$\text{span } V_K \leq c(K) \leq n.$$

Putting  $\text{span } V_K = n$  and  $\text{span } V_K \leq c(K) \leq n$  together shows that  $\text{span } V_K = c(K)$ . (Incidentally, we have also shown that  $c(K) = n$ , which will be useful for the next problem.)

- 4.8.3. **Problem:** (5) Determine the crossing number  $c(K)$  of the knot  $K$  depicted by the diagram  $D$  in Figure 4.7. Justify your answer.



Figure 4.7:  $D$ , crazy knot diagram

**Solution:** The answer is 11.

Observe that  $D$  is reduced alternating and has  $n = 11$  crossings.

By Problem 4.7.3, we know  $\text{span} \langle D \rangle = 4n$ , which gives

$$\text{span } V_K = n.$$

By Problem 4.8.1, we have

$$\text{span } V_K \leq c(K) \leq n.$$

Putting  $\text{span } V_K = n$  and  $\text{span } V_K \leq c(K) \leq n$  together shows that  $c(K) = n = 11$ .

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