PUMaC 2013

Team Round

RULES: Answer as many of the following 16 questions as you can. In the 4×4 square below, if you get an answer correct, an O will be added in the appropriate square. If you get it wrong, an X will be added.

Your total score will be

$$\#\text{O's} + 2\#(\text{four O's in a row, column or diagonal}) - 2\#(\text{four X's in a row, column or diagonal}).$$

Good luck! You have 30 minutes.

The square:

3	9	6	4
1	11	7	15
2	10	13	12
14	5	16	8

1. A token is placed in the leftmost square in a strip of four squares. In each move, you are allowed to move the token left or right along the strip by sliding it a single square, provided that the token stays on the strip. In how many ways can the token be moved so that after exactly 15 moves, it is in the rightmost square of the strip?

2. (Following question 1) Now instead consider an infinite strip of squares, labeled with the integers 0, 1, 2, ... in that order. You start at the square labeled 0. You want to end up at the square labeled 3. In how many ways can this be done in exactly 15 moves?

3. The area of a circle centered at the origin, which is inscribed in the parabola $y = x^2 - 25$, can be expressed as $\frac{a}{b}\pi$, where a and b are coprime positive integers. What is the value of $a + b$?

4. Find the sum of all positive integers m such that 2^m can be expressed as a sum of four factorials (of positive integers).

   Note: The factorials do not have to be distinct. For example, $2^4 = 16$ counts, because it equals $3! + 3! + 2! + 2!$.

5. A palindrome number is a positive integer that reads the same forward and backward. For example, 1221 and 8 are palindrome numbers whereas 69 and 157 are not. A and B are 4-digit palindrome numbers. C is a 3-digit palindrome number. Given that $A - B = C$, what is the value of C?

6. How many positive integers n less than 1000 have the property that the number of positive integers less than n which are coprime to n is exactly $\frac{n}{3}$?

7. Find the total number of triples of integers (x, y, n) satisfying the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{n^2}$, where n is either 2012 or 2013.

8. Let k be a positive integer with the following property: For every subset A of $\{1, 2, ..., 25\}$ with $|A| = k$, we can find distinct elements x and y of A such that $\frac{2}{3} \leq \frac{x}{y} \leq \frac{3}{2}$. Find the smallest possible value of k.

9. If two distinct integers from 1 to 50 inclusive are chosen at random, what is the expected value of their product? Note: The expectation is defined as the sum of the products of probability and value, i.e., the expected value of a coin flip that gives you \$10 if head and \$5 if tail is $\frac{1}{2} \times \$10 + \frac{1}{2} \times \$5 = \$7.5$.

10. On a plane, there are 7 seats. Each is assigned to a passenger. The passengers walk on the plane one at a time. The first passenger sits in the wrong seat (someone else's). For all the following people, they either sit in their assigned seat, or if it is full, randomly pick another. You are the last person to board the plane. What is the probability that you sit in your own seat?'

11. If two points are selected at random on a fixed circle and the chord between the two points is drawn, what is the probability that its length exceeds the radius of the circle?

12. Let D be a point on the side BC of $\triangle ABC$. If $AB = 8$, $AC = 7$, $BD = 2$ and $CD = 1$, find AD.

13. The equation $x^5 - 2x^4 - 1 = 0$ has five complex roots r_1, r_2, r_3, r_4, r_5. Find the value of

$$\frac{1}{r_1^8} + \frac{1}{r_2^8} + \frac{1}{r_3^8} + \frac{1}{r_4^8} + \frac{1}{r_5^8}.$$

14. Shuffle a deck of 71 playing cards which contains 6 aces. Then turn up cards from the top until you see an ace. What is the average number of cards required to be turned up to find the first ace?

15. Prove:

$$|\sin a_1| + |\sin a_2| + |\sin a_3| + ... + |\sin a_n| + |\cos(a_1 + a_2 + a_3 + ... + a_n)| \geq 1.$$

16. Is $\cos 1^\circ$ rational? Prove.