PUMaC 2013

Algebra B

1. [**3**] Suppose $a, b, c > 0$ are integers such that

$$abc - bc - ac - ab + a + b + c = 2013.$$

Find the number of possibilities for the ordered triple (a, b, c).

2. [**3**] Betty Lou and Peggy Sue take turns flipping switches on a 100×100 grid. Initially, all switches are "off." Betty Lou always flips a horizontal row of switches on her turn; Peggy Sue always flips a vertical column of switches. When they finish, there is an *odd* number of switches turned "on" in each row and column. Find the maximum number of switches that can be on, in total, when they finish.

3. [**4**] Let $x_1 = 1/20$, $x_2 = 1/13$, and

$$x_{n+2} = \frac{2x_n x_{n+1}(x_n + x_{n+1})}{x_n^2 + x_{n+1}^2}$$

for all integers $n \geq 1$. Evaluate $\sum_{n=1}^{\infty}(1/(x_n + x_{n+1}))$.

4. [**4**] Let $f(x) = 1 - |x|$. Let

$$f_n(x) = (\overbrace{f \circ \cdots \circ f}^{n \text{ copies}})(x)$$
$$g_n(x) = |n - |x||$$

Determine the area of the region bounded by the x-axis and the graph of the function $\sum_{n=1}^{10} f_n(x) + \sum_{n=1}^{10} g_n(x)$.

5. [**5**] Find the number of pairs (n, C) of positive integers such that $C \leq 100$ and $n^2 + n + C$ is a perfect square.

6. [**6**] Suppose a, b are nonzero integers such that two roots of $x^3 + ax^2 + bx + 9a$ coincide, and all three roots are integers. Find $|ab|$.

7. [**7**] Evaluate

$$\sqrt{2013 + 276\sqrt{2027 + 278\sqrt{2041 + 280\sqrt{2055 + \ldots}}}}$$

8. [**8**] If x, y are real, then the *absolute value* of the complex number $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}.$$

Find the number of polynomials $f(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + t^4$ such that $A_0, \ldots, A_3$ are integers and all roots of f in the complex plane have absolute value ≤ 1.