Algebra B Solutions

B1

Suppose $a, b, c > 0$ are integers such that

$$abc - bc - ac - ab + a + b + c = 2013. \tag{B.1}$$

Find the number of possibilities for the ordered triple (a, b, c).

Solution. Subtracting 1 from both sides and factoring, we obtain

$$2012 = (a-1)(b-1)(c-1). \tag{B.2}$$

Writing $a' = a - 1$ and similarly with b', c', it suffices to count the ordered integer triples (a', b', c') such that $a', b', c' \geq 0$ and $a'b'c' = 2012$.

Immediately, we require $a', b', c' > 0$. Let $\tau(n)$ denote the number of positive divisors of n. The prime factorization of 2012 is $2^2 \cdot 503^1$, so the number of possible triples is

$$\sum_{\substack{d \mid 2012 \\ d>0}} \tau(d) = \tau(1) + \tau(2) + \tau(2^2) + \tau(503) + \tau(2 \cdot 503) + \tau(2^2 \cdot 503)$$

which simplifies to $1 + 2 + 3 + 2 + 4 + 6 = \boxed{18}$. □

B2

Betty Lou and Peggy Sue take turns flipping switches on a 100×100 grid. Initially, all switches are "off." Betty Lou always flips a horizontal row of switches on her turn; Peggy Sue always flips a vertical column of switches. When they finish, there is an *odd* number of switches turned "on" in each row and column. Find the maximum number of switches that can be on, in total, when they finish.

Remark. The names are taken from the Beach Boys song "Barbara Ann."

Solution. We can assume Betty Lou flips each row at most once, and similarly with Peggy Sue and the columns. After permuting the rows and columns, we can assume all the flipped rows are adjacent to each other, and all the flipped columns are adjacent to each other. Therefore, if R is the number of flipped rows and C the number of flipped columns when they finish, then the total number of "on" switches is

$$100R + 100C - 2RC = 2(50R + 50C - RC). \tag{B.3}$$

We want to maximize this quantity given that $R, C \in [1, 100]$ are both odd. But this is equivalent to *minimizing* the quantity

$$RC - 50R - 50C = (R - 50)(C - 50) - 50^2 \tag{B.4}$$

under the same conditions. We do so when either $(R, C) = (1, 99)$ or $(R, C) = (99, 1)$. So the maximum total number of "on" switches is $100 \cdot 1 + 100 \cdot 99 - 2 \cdot 1 \cdot 99 = \boxed{9802}$. $\square$

B3

Let $x_1 = 1/20$, $x_2 = 1/13$, and

$$x_{n+2} = \frac{2x_n x_{n+1}(x_n + x_{n+1})}{x_n^2 + x_{n+1}^2} \tag{B.5}$$

for all integers $n \geq 1$. Evaluate $\sum_{n=1}^{\infty}(1/(x_n + x_{n+1}))$.

Solution. The relation implies

$$\frac{1}{x_{n+2}} = \frac{x_n^2 + x_{n+1}^2}{2x_n x_{n+1}(x_n + x_{n+1})} = \frac{1}{2x_n} + \frac{1}{2x_{n+1}} - \frac{1}{x_n + x_{n+1}}. \tag{B.6}$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{x_n + x_{n+1}} = \sum_{n=1}^{\infty}\left(\frac{1}{2x_n} + \frac{1}{2x_{n+1}} - \frac{1}{x_{n+2}}\right) &= \sum_{n=1}^{\infty}\frac{1}{2x_n} + \sum_{n=2}^{\infty}\frac{1}{2x_n} - \sum_{n=3}^{\infty}\frac{1}{x_n} \\ &= \frac{1}{2x_1} + \frac{1}{x_2} \\ &= \boxed{23}. \end{aligned} \tag{B.7}$$

Above, the convergence of the series $\sum_{n=1}^{\infty}(1/x_n)$ is assured because $x_{n+2} \geq 2x_n$. $\square$

B4

Let $f(x) = 1 - |x|$. Let

$$f_n(x) = (\overbrace{f \circ \cdots \circ f}^{n \text{ copies}})(x) \tag{B.8}$$

$$g_n(x) = |n - |x|| \tag{B.9}$$

Determine the area of the region bounded by the x-axis and the graph of the function $\sum_{n=1}^{10} f_n(x) + \sum_{n=1}^{10} g_n(x)$.

Solution. First, $f_n(x) + g_n(x) = 0$ for all x outside the interval $[-n, +n]$. Within that interval, $f_n(x), g_n(x)$ are everywhere nonnegative. Specifically, the region bounded by the x-axis and $f_n(x)$ looks like a row of n adjacent isosceles triangles, each having height 1 and base of length 2 along the x-axis. The region bounded by the x-axis and $g_n(x)$ looks like a single isosceles triangle of height n and base of length $2n$ along the x-axis.

Therefore, $F_n(x) = f_n(x) + g_n(x)$ is nonnegative for all n and x, and the area of the region between the x-axis and $F_n(x)$ is $n^2 + n$. As the nonnegativity implies the areas under the various F_n are cumulative, we compute

$$\sum_{n=1}^{10}(n^2+n) = \sum_{n=1}^{10} n^2 + \sum_{n=1}^{10} n = \frac{10(1+10)(1+2\cdot 10)}{6} + \frac{10(1+10)}{2} = 385 + 55 \quad \text{(B.10)}$$

to get an answer of $\boxed{440}$. □

B5

Find the number of pairs (n, C) of positive integers such that $C \leq 100$ and $n^2 + n + C$ is a perfect square.

Remark. This problem was inspired by the "New Year's Problem" of the 2008-2009 Wisconsin Math Talent Search.

Solution. We can write $n^2 + n + C = (n+m)^2$ for some integer $m \geq 1$. Expanding and rearranging, $C = m^2 + (2m-1)n$, so we have reduced the problem to counting pairs (m, n) of positive integers such that $m^2 + (2m-1)n \leq 100$. This is

$$\sum_{m=1}^{10} \left\lfloor \frac{100 - m^2}{2m-1} \right\rfloor = 99 + 32 + 18 + 12 + 8 + 5 + 3 + 2 + 1 + 0 \quad \text{(B.11)}$$

which simplifes to $\boxed{180}$. □

B6

Suppose a, b are nonzero integers such that two roots of $x^3 + ax^2 + bx + 9a$ coincide, and all three roots are integers. Find $|ab|$.

Solution. Write

$$x^3 + ax^2 + bx + 9a = (x-r)^2(x-s) \quad \text{(B.12)}$$

for some integers r, s. Expanding and equating coefficients, we require

$$\begin{cases} r^2 s & = -9a \\ r^2 + 2rs & = b \\ 2r + s & = -a \end{cases} \tag{B.13}$$

Combining the first and third constraints gives $r^2 s - 18r - 9s = 0$. Solving this quadratic in r, we find that $36(s^2 + 9)$ must be a perfect square, because r, s are integers. Therefore, $s^2 + 9 = d^2$ for some $d > 0$, whence $(d + s)(d - s) = 9$. We cannot have $s = 0$ because this would imply $a = 0$ by the first constraint in the display, so the only solutions for s are ± 4. Then $(r, s) = \pm(6, 4)$, so $|ab| = |(2r + s)(r^2 + 2rs)| = \boxed{1344}$ in both cases. □

B7

Evaluate

$$\sqrt{2013 + 276\sqrt{2027 + 278\sqrt{2041 + 280\sqrt{2055 + \ldots}}}} \tag{B.14}$$

Remark. This problem was inspired by a theorem of Ramanujan. He famously posed the similar problem of evaluating

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \ldots}}}} \tag{B.15}$$

in the *Journal of the Indian Mathematical Society*, with no response after six months. A good, albeit somewhat romanticized, place to read about Ramanujan is [Ka].

Solution. We prove that

$$\begin{aligned} & a + N + n \\ &= \sqrt{(a+n)^2 + aN + N\sqrt{(a+n)^2 + a(N+n) + (N+n)\sqrt{(a+n)^2 + \ldots}}} \end{aligned} \tag{B.16}$$

for all $a \in \mathbb{R}$ and $N, n \in \mathbb{Z}_{\geq 0}$ with $n \mid N$. The result holds for $N = 0$ and any a, n, and if it holds for an arbitrary triple (a, N, n), then squaring both sides and simplifying,

$$N + a + 2n = \sqrt{(a+n)^2 + a(N+n) + (N+n)\sqrt{(a+n)^2 + \ldots}} \tag{B.17}$$

so it holds for $(a, N + n, n)$, whence induction finishes the proof.

Our original problem is the case $(a, N, n) = (7, 276, 2)$. The answer is $a + N + n = \boxed{285}$. □

B8

If x, y are real, then the *absolute value* of the complex number $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}. \tag{B.18}$$

Find the number of polynomials $f(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + t^4$ such that $A_0, \ldots, A_3$ are integers and all roots of f in the complex plane have absolute value ≤ 1.

Solution. Since complex conjugation distributes over addition and multiplication, $x + iy$ is a root of f if and only if $x - iy$ is also a root of f. In particular, f has exactly 0, 2, or 4 real roots. Before dealing with each case, let us describe the roots in more detail:

We check that $|z_1 z_2| = |z_1||z_2|$ for all complex z_1, z_2. Given that the coefficients of f are integers, it follows that the product of the nonzero roots of f must be a nonzero integer of absolute value ≤ 1, hence equal to 1. But each *individual* root of f has absolute value ≤ 1. So each root of f is either 0 or has absolute value 1. In particular, the real roots of f are 0 or ± 1. If $x + iy$ is a nonreal root of f, then $-1 < x < +1$ and

$$\begin{aligned}(t - (x + iy))(t - (x - iy)) &= t^2 - 2xt + (x^2 + y^2) \\ &= t^2 - 2xt + 1\end{aligned} \tag{B.19}$$

occurs in the linear factorization above. Now the case analysis:

1. Suppose there are 0 real roots. Then $f = (t^2 - 2x_1 t + 1)(t^2 - 2x_2 t + 1)$ for some $x_1, x_2 \in (-1, +1)$ such that $4x_1 x_2$ and $2x_1 + 2x_2$ are both integers, and any such x_1, x_2 will do. Therefore, the two-way map

   $$(t^2 - 2x_1 t + 1)(t^2 - 2x_2 t + 1) \leftrightarrow (u - 2x_1)(u - 2x_2) \tag{B.20}$$

   is a bijection between possibilities for f and polynomials $g(u) = u^2 + bu + c$ such that b, c are integers and the roots of g belong to $(-2, +2)$. By the quadratic formula, the condition on the roots works out to $b - 4 < \pm\sqrt{b^2 - 4c} < b + 4$, meaning $b^2 \geq 4c$ and $-2b - 4 < c < +2b - 4$. But we also know $b, c \in (-4, +4)$ from expressing b, c in terms of x_1, x_2. Altogether, the only possibilities for (b, c) are

   $$\begin{aligned}&(0, 0), \\ &(1, -3), \\ &(2, -3), (2, -2), (2, -1), \\ &(3, -3), (3, -2), (3, -1), (3, 0), (3, 1),\end{aligned} \tag{B.21}$$

   a total of 10 possibilities for f.

2. If there are 2 real roots, then there are 6 possibilities for their product. There are 3 possibilities for the remaining $t^2 - 2xt + 1$ factor, namely $x \in \{0, \pm 1/2\}$. Thus there are 18 possibilities for f.

3. If all 4 roots are real, then there are

$$\binom{3+4-1}{4} = 15 \tag{B.22}$$

possibilities for f, either by brute-force enumeration or the lemma below.

Altogether there are $10 + 18 + 15 = \boxed{43}$ possibilities. □

Lemma. *Let $r(n, N)$ be the number of ways of choosing N things from n distinguished choices, allowing repetition in choice but not distinguishing order of choosing. Then*

$$r(n, N) = \binom{n+N-1}{N}. \tag{B.23}$$

Proof. Imagine having N stars in a row and $n-1$ vertical bars to place between them. We claim there is a bijection between the possible placements of the bars relative to the horizontal row of stars—where a single bar can be placed either left of the stars, right of the stars, or in between two adjacent stars, and multiple bars can occupy the same position—and the number of ways of choosing N things from n distinguished choices, allowing repetition but not distinguishing choice order. For, add an "immobile" bar to the right of the stars. Now match the n bars with the choices. Let the number of times a given choice was picked equal the number of stars between its bar and the bar immediately to the left. It is straightforward to verify that this is a bijection.

Thinking of the bars, sans the immobile one, together with the stars as a row of $n + N - 1$ objects, the number of allowed placements of the free-moving bars is exactly the number of ways of choosing N things at all once from $n + N - 1$ choices, as needed. □

Remark. The proof above has led to this lemma being called "Stars and Bars." That this is also the nickname of the flag of the defunct Confederate States of America is purely coincidental!

References

[Ka] R. Kanigel. *The Man Who Knew Infinity: a Life of the Genius Ramanujan.* New York: Charles Scribner's Sons (1991). ISBN 0-684-19259-4.