PUMaC 2013

Combinatorics B

1. [**3**] Including the original, how many ways are there to rearrange the letters in PRINCETON so that no two vowels (I, E, O) are consecutive and no three consonants (P, R, N, C, T, N) are consecutive?

2. [**3**] The number of positive integer pairs (a, b) that have a dividing b and b dividing 2013^{2014} can be written as $2013n + k$, where n and k are integers and $0 \leq k < 2013$. What is k? Recall $2013 = 3 \cdot 11 \cdot 61$.

3. [**4**] Chris's pet tiger travels by jumping north and east. Chris wants to ride his tiger from Fine Hall to McCosh, which is 3 jumps east and 10 jumps north. However, Chris wants to avoid the horde of PUMaC competitors eating lunch at Frist, located 2 jumps east and 4 jumps north of Fine Hall. How many ways can he get to McCosh without going through Frist?

4. [**4**] Mereduth has many red boxes and many blue boxes. Coloon has placed five green boxes in a row on the ground, and Mereduth wants to arrange some number of her boxes on top of his row. Assume that each box must be placed so that it straddles two lower boxes. Including the one with no boxes, how many arrangements can Mereduth make?

5. [**5**] A sequence of vertices $v_1, v_2, \ldots, v_k$ in a graph, where $v_i = v_j$ only if $i = j$ and k can be any positive integer, is called a *cycle* if v_1 is attached by an edge to v_2, v_2 to v_3, and so on to v_k connected to v_1. Rotations and reflections are distinct: A, B, C is distinct from A, C, B and B, C, A. Suppose a simple graph G has 2013 vertices and 3013 edges. What is the minimal number of cycles possible in G?

6. [**6**] An integer sequence $a_1, a_2, \ldots, a_n$ has $a_1 = 0$, $a_n \leq 10$ and $a_{i+1} - a_i \geq 2$ for $1 \leq i < n$. How many possibilities are there for this sequence? The sequence may be of any length.

7. [**7**] You are eating at a fancy restaurant with a person you wish to impress. For some reason, you think that eating at least one spicy course and one meat-filled course will impress the person. The meal is five courses, with four options for each course. Each course has one option that is spicy and meat-filled, one option that is just spicy, one that is just meat-filled, and one that is neither spicy nor meat-filled. How many possible meals can you have?

8. [**8**] You roll three fair six-sided dice. Given that the highest number you rolled is a 5, the expected value of the sum of the three dice can be written as $\frac{a}{b}$ in simplest form. Find $a + b$.