PUMaC 2013

Combinatorics B Solutions

1. [**3**] Including the original, how many ways are there to rearrange the letters in PRINCETON so that no two vowels (I, E, O) are consecutive and no three consonants (P, R, N, C, T, N) are consecutive?

   Solution We have six consonants and three vowels. The consonants then occur in at most four segments sectioned off by vowels. If only three segments have consonants in them, the only possibilities are CCVCCVCCV and VCCVCCVCC. If all four have consonants, we have four choose two, or six, ways to add the last two consonants. In all, we have $6+2=8$ layouts, and $\frac{6!}{2} \times 3! = 360(6) = 2160$ ways to fill the layouts with letters. So the total is $2160(8) = 17280$.

2. [**3**] The number of positive integer pairs (a, b) that have a dividing b and b dividing 2013^{2014} can be written as $2013n + k$, where n and k are integers and $0 \leq k < 2013$. What is k? Recall $2013 = 3 \cdot 11 \cdot 61$.

   Solution This is equivalent to choosing $x_1, x_2, x_3, y_1, y_2, y_3$ nonnegative integers less than or equal to 2014 so $x_i \leq y_i$. There are $2015 + 2015(2014)/2 \equiv 2 + 1 = 3$ ways to choose each x_i, y_i pair, for a total of $27(2013)$ ways.

3. [**4**] Chris's pet tiger travels by jumping north and east. Chris wants to ride his tiger from Fine Hall to McCosh, which is 3 jumps east and 10 jumps north. However, Chris wants to avoid the horde of PUMaC competitors eating lunch at Frist, located 2 jumps east and 4 jumps north of Fine Hall. How many ways can he get to McCosh without going through Frist?

   Solution There are 13 choose 3 ways to get to McCosh without the restriction, or $13 * 11 * 2 = 286$. For Chris to avoid lunchers, he needs to not take a way through Frist; there are 6 choose 2, or 15, ways to Frist, and 7 choose 1, or 7, ways from it. This gives a total of 105 bad paths, for a final total of 181 paths to McCosh.

4. [**4**] Mereduth has many red boxes and many blue boxes. Coloon has placed five green boxes in a row on the ground, and Mereduth wants to arrange some number of her boxes on top of his row. Assume that each box must be placed so that it straddles two lower boxes. Including the one with no boxes, how many arrangements can Mereduth make?

   Solution Let K_n be the number of arrangements we can make on n boxes. On top of the n green boxes, Mereduth has some string of boxes with no gaps, possibly of length zero. If this is of length i, we find that the number of further arrangements that can be made is $2^i K_i K_{n-i-1}$. Then we have $K_0 = 1$, $K_1 = 1$, $K_2 = 2K_1K_0 + K_0K_1 = 3$, $K_3 = 4K_2K_0 + 2K_1^2 + K_0K_2 = 4(3) + 2 + 3 = 17$, $K_4 = 8K_3K_0 + 4K_2K_1 + 2K_1K_2 + K_0K_3 = 9(17) + 6(3) = 171$, and

   $$K_5 = 17 \times 171 + 10 \times 17 + 4 \times 3^2 = 2907 + 160 + 36 = 3113$$

   So there are 3113 rearrangements.

5. [**5**] A sequence of vertices $v_1, v_2, \ldots, v_k$ in a graph, where $v_i = v_j$ only if $i = j$ and k can be any positive integer, is called a *cycle* if v_1 is attached by an edge to v_2, v_2 to v_3, and so on to v_k connected to v_1. Rotations and reflections are distinct: A, B, C is distinct from A, C, B and B, C, A. Suppose a simple graph G has 2013 vertices and 3013 edges. What is the minimal number of cycles possible in G?

Solution We assume G is connected. Then 2012 edges make a spanning subtree of G. Adding any edge is going to allow a new loop; if this loop involves n vertices, $2n$ cycles result. Each new edge is going to be involved in at least six new cycles, then, and it can be seen pretty easily that this bound is attained. So the minimum is 6006.

6. [**6**] An integer sequence $a_1, a_2, \ldots, a_n$ has $a_1 = 0$, $a_n \leq 10$ and $a_{i+1} - a_i \geq 2$ for $1 \leq i < n$. How many possibilities are there for this sequence? The sequence may be of any length.

   Solution Suppose K_d counts the number of sequences with $a_n \leq d$. There are K_{d-2} sequences with $a_n = d$ and K_{d-1} with $a_n < d$, so we have $K_d = K_{d-1} + K_{d-2}$. Lovely; we find $K_0 = 1$, $K_1 = 1$, and from there get $K_2 = 2$, $K_3 = 3$, $K_4 = 5$, $K_5 = 8$, $K_6 = 13$, $K_7 = 21$, $K_8 = 34$, $K_9 = 55$, and $K_10 = 89$. So the answer is 89.

7. [**7**] You are eating at a fancy restaurant with a person you wish to impress. For some reason, you think that eating at least one spicy course and one meat-filled course will impress the person. The meal is five courses, with four options for each course. Each course has one option that is spicy and meat-filled, one option that is just spicy, one that is just meat-filled, and one that is neither spicy nor meat-filled. How many possible meals can you have?

   Solution Inclusion exclusion: there are 4^5 meals if we ignore requirements. There are 2^5 that avoid spice, 2^5 that avoid meat, and one that avoids both, for $4^5 - 2(2^5) + 1 = 961$ total meals.

8. [**8**] You roll three fair six-sided dice. Given that the highest number you rolled is a 5, the expected value of the sum of the three dice can be written as $\frac{a}{b}$ in simplest form. Find $a + b$.

   Solution Note that this is *not* the same as fixing one die at 5, then randomizing the rest from 1 to 5. To compute the expected value correctly, we consider three cases: rolls that include exactly one, two, three 5s. To make counting easier we think of the rolls as happening sequentially and count each distinct sequence as an instance. Non-5 rolls are equally likely to be 1, 2, 3, 4 and are therefore worth 2.5 on average. So we have:

   Three 5s: $(1 * 4^0) * 15 = 15$

   Two 5s: $(3 * 4^1) * 12.5 = 150$

   One 5: $(3 * 4^2) * 10.0 = 480$

   We end up with $1 + 12 + 48 = 61$ instances and a total sum of 645. The fraction $\frac{645}{61}$, already in simplest form, gives sum 706, and that is the answer.