PUMaC 2013

Geometry B

1. [**3**] We construct three circles: O with diameter AB and area $12 + 2x$, P with diameter AC and area $24 + x$, and Q with diameter BC and area $108 - x$. Given that C is on circle O, compute x.

2. [**3**] Triangle ABC satisfies $\angle ABC = \angle ACB = 78^\circ$. Points D and E lie on AB, AC and satisfy $\angle BCD = 24^\circ$ and $\angle CBE = 51^\circ$. If $\angle BED = x^\circ$, find x.

3. [**4**] Consider all planes through the center of a $2 \times 2 \times 2$ cube that create cross sections that are regular polygons. The sum of the cross sections for each of these planes can be written in the form $a\sqrt{b} + c$, where b is a square-free positive integer. Find abc.

4. [**4**] An equilateral triangle is given. A point lies on the incircle of the triangle. The smallest two distances from the point to the sides of the triangle is 1 and 4. The sidelength of this triangle can be expressed as $\frac{a\sqrt{b}}{c}$ where $(a, c) = 1$ and b is not divisible by the square of an integer greater than 1. Find $a + b + c$.

5. [**5**] Circle w with center O meets circle Γ at X, Y, and O is on Γ. Point $Z \in \Gamma$ lies outside w such that $XZ = 11$, $OZ = 15$ and $YZ = 13$. If the radius of circle w is r, find r^2.

6. [**6**] Draw an equilateral triangle with center O. Rotate the equilateral triangle $30^\circ, 60^\circ, 90^\circ$ with respect to O so there would be four congruent equilateral triangles on each other. Look at the diagram. If the smallest triangle has area 1, the area of the original equilateral triangle could be expressed as $p + q\sqrt{r}$ where p, q, r are positive integers and r is not divisible by a square greater than 1. Find $p + q + r$.

7. [**7**] A tetrahedron $ABCD$ satisfies $AB = 6$, $CD = 8$, and $BC = DA = 5$. Let V be the maximum volume of $ABCD$ possible. If we can write $V^4 = 2^n 3^m$ for some integers n and m, find mn.

8. [**8**] Triangle $A_1B_1C_1$ is an equilateral triangle with sidelength 1. For each $n > 1$, we construct triangle $A_nB_nC_n$ from $A_{n-1}B_{n-1}C_{n-1}$ according to the following rule: A_n, B_n, C_n are points on segments $A_{n-1}B_{n-1}$, $B_{n-1}C_{n-1}$, $C_{n-1}A_{n-1}$ respectively, and satisfy the following:

$$\frac{A_{n-1}A_n}{A_nB_{n-1}} = \frac{B_{n-1}B_n}{B_nC_{n-1}} = \frac{C_{n-1}C_n}{C_nA_{n-1}} = \frac{1}{n-1}$$

So for example, $A_2B_2C_2$ is formed by taking the midpoints of the sides of $A_1B_1C_1$. Now, we can write $\frac{|A_5B_5C_5|}{|A_1B_1C_1|} = \frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$. (For a triangle $\triangle ABC, |ABC|$ denotes its area.)