



## Number Theory B Solutions

1. [3] If  $p, q$  and  $r$  are primes with  $pqr = 7(p + q + r)$ , find  $p + q + r$ .

**Solution** Without loss of generality, we see that we must have  $p = 7$ . Next,  $qr = 7 + q + r$  iff  $(q - 1)(r - 1) = 8$ . The only prime solution is  $(q, r) = (3, 5)$  up to permutation. Then  $p + q + r = 15$ .

2. [3] What is the smallest positive integer  $n$  such that  $2013^n$  ends in 001 (i.e. the rightmost three digits of  $2013^n$  are 001)?

**Solution** Firstly,

$$\begin{aligned} 2013^{100} &\equiv 13^{100} \\ &= (10 + 3)^{100} \\ &\equiv 3^{100} \\ &\equiv 49^{10} \\ &\equiv 401^5 \\ &= (400 + 1)^5 \\ &\equiv 1 \pmod{1000} \end{aligned}$$

yields  $n|100$ . Next

$$\begin{aligned} 2013^{50} &\equiv (10 + 3)^{50} \\ &\equiv 3^{50} + 50 \cdot 10 \cdot 3^{49} + \binom{50}{2} \cdot 10^2 \cdot 3^{48} \\ &\equiv 3^{50} \\ &\equiv 49^5 \\ &\equiv 807^2 \\ &\equiv 249 \pmod{1000} \end{aligned}$$

and

$$\begin{aligned} 2013^{20} &\equiv (10 + 3)^{20} \\ &\equiv 3^{20} + 20 \cdot 10 \cdot 3^{19} \\ &\equiv 49^2 + 400 \\ &\equiv 801 \pmod{1000} \end{aligned}$$

shows that  $n = 100$ .

3. [4] Find the smallest positive integer  $x$  such that

- $x$  is 1 more than a multiple of 3,
- $x$  is 3 more than a multiple of 5,



- $x$  is 5 more than a multiple of 7,
- $x$  is 9 more than a multiple of 11, and
- $x$  is 2 more than a multiple of 13.

**Solution** The first four conditions is the same as  $x \equiv -2 \pmod{1155}$ . By trial, the answer is  $x = 12703$ .

Alternatively, we have to solve the two equations  $x \equiv -2 \pmod{1155}$  and  $x \equiv 2 \pmod{13}$ . Thus we can write  $x = 1155y - 2$ , and substituting this into the second equation, we obtain:  $1155y - 2 \equiv 11y - 2 \equiv 2 \pmod{13} \Rightarrow -2y \equiv 4 \pmod{13} \Rightarrow y \equiv -2 \equiv 11 \pmod{13}$ , so  $x = 11 \cdot 1155 - 2 = 12703$

4. [4] Compute the smallest integer  $n \geq 4$  such that  $\binom{n}{4}$  ends in 4 or more zeroes (i.e. the rightmost four digits of  $\binom{n}{4}$  are 0000).

**Solution** Firstly,  $2^4 | \binom{n}{4}$  iff  $2^7 | n(n-1)(n-2)(n-3)$  if and only if  $n \equiv 0, 1, 2, 3 \pmod{64}$ . Next,  $5^4 | \binom{n}{4}$  if and only if  $n \equiv 0, 1, 2, 3 \pmod{625}$ . By trial, the smallest  $n$  is 8128.

5. [5] Let  $A$  be the greatest possible value of a product of positive integers that sums to 2014. Compute the sum of all bases and exponents in the prime factorization of  $A$ . For example, if  $A = 7 \cdot 11^5$ , the answer would be  $7 + 11 + 5 = 23$ .

**Solution** Note that if we have a large enough  $n$ , odd  $n \geq 5$ , break it as  $n = \frac{n-1}{2} + \frac{n+1}{2}$ . This product is larger than  $n$ . For even  $n \geq 4$ , break it as  $n = \frac{n}{2} + \frac{n}{2}$ . This product is larger than  $n$ . Furthermore, noting that  $2^3 < 3^2$ , we should have at most two 2's. So the optimum case is  $(2)(2) + (670)(3) = 2014$ . The answer is 677.

6. [6] Let  $d$  be the greatest common divisor of  $2^{30^{10}} - 2$  and  $2^{30^{45}} - 2$ . Find the remainder when  $d$  is divided by 2013.

**Solution** We have

$$d = (2^{30^{10}} - 2, 2^{30^{45}} - 2) = 2 \cdot (2^{(30^{10}-1, 30^{45}-1)} - 1) = 2 \cdot (2^{30^{(10,45)}-1} - 1) = 2^{30^5} - 2.$$

As  $\phi(2013) = 1200$  and  $1200 | 30^5$ , the remainder is 2012.

7. [7] Define a "digitized number" as a ten-digit number  $a_0a_1 \dots a_9$  such that for  $k = 0, 1, \dots, 9$ ,  $a_k$  is equal to the number of times the digit  $k$  occurs in the number. Find the sum of all digitized numbers.

**Solution** From the condition, we need

$$\sum_{k=0}^9 ka_k = \sum_{k=0}^9 a_k = 10.$$

In particular,

$$a_0 = \sum_{k=2}^9 (k-1)a_k \geq \sum_{k=2}^{9-a_0} (k-1)$$

yields  $a_0 \geq 6$ . By examining each case, we find the only solution 6210001000.



8. [8] What is the largest positive integer that cannot be expressed as a sum of non-negative integer multiples of 13, 17 and 23?

**Solution** There are numerous approaches to this problem, and no approach that attempts to find the last obtained remainder modulo any of the three numbers in sums of them will fail. The below is our approach:

By trial, the following gives the smallest positive integer in the form  $17a + 23b$  for integers  $a, b \geq 0$  in different residue classes modulo 13 (starting from the class 0):

$$91, 40, 80, 68, 17, 57, 97, 46, 34, 74, 23, 63, 51.$$

(Indeed, one should note that  $5(17) \equiv 2(23) \pmod{13}$  and  $3(23) \equiv 17 \pmod{13}$  to reduce the number of cases for consideration.)

The largest  $n$  which is not of the form  $13x + 17y + 23z$  is therefore  $97 - 13 = 84$ .