PUMaC 2013

Individual Finals A

1. Prove that
$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\le\frac{1}{6ab+c^2}+\frac{1}{6bc+a^2}+\frac{1}{6ca+b^2}$$
for any positive real numbers a, b and c satisfying $a^2+b^2+c^2=1$.

2. Let γ be the incircle of $\triangle ABC$ (i.e. the circle inscribed in $\triangle ABC$) and I be the center of γ. Let D, E and F be the feet of the perpendiculars from I to BC, CA and AB respectively. Let D' be the point on γ such that DD' is a diameter of γ. Suppose the tangent to γ through D intersects the line EF at P. Suppose the tangent to γ through D' intersects the line EF at Q. Prove that $\angle PIQ+\angle DAD'=180°$.

3. A *graph* consists of a set of vertices, some of which are connected by (undirected) edges. A *star* of a graph is a set of edges with a common endpoint. A *matching* of a graph is a set of edges such that no two have a common endpoint. Show that if the number of edges of a graph G is larger than $2(k-1)^2$, then G contains a matching of size k or a star of size k.