Individual Finals A Solutions

1. Prove that
$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\le\frac{1}{6ab+c^2}+\frac{1}{6bc+a^2}+\frac{1}{6ca+b^2}$$
for any positive real numbers a, b and c satisfying $a^2+b^2+c^2=1$.

Solution

By the AM-GM inequality, $3a^2+3b^2\ge 6ab$, $3b^2+3c^2\ge 6bc$ and $3c^2+3a^2\ge 6ca$. Hence
$$\frac{1}{3a^2+3b^2+c^2}+\frac{1}{3b^2+3c^2+a^2}+\frac{1}{3c^2+3a^2+b^2}\le\frac{1}{6ab+c^2}+\frac{1}{6bc+a^2}+\frac{1}{6ca+b^2}.$$
Now
$$\begin{aligned}\sum_{cyc}\frac{1}{3b^2+3c^2+a^2}-\sum_{cyc}\frac{1}{a^2+2}&=\sum_{cyc}\left(\frac{1}{3-2a^2}-\frac{1}{a^2+2}\right)\\&=\sum_{cyc}\frac{3a^2-1}{(3-2a^2)(2+a^2)}\\&=\sum_{cyc}\left(\frac{a^2-b^2}{(3-2a^2)(2+a^2)}-\frac{c^2-a^2}{(3-2a^2)(2+a^2)}\right)\\&=\sum_{cyc}\left(\frac{a^2-b^2}{(3-2a^2)(2+a^2)}-\frac{a^2-b^2}{(3-2b^2)(2+b^2)}\right)\\&=\sum_{cyc}\frac{(a-b)^2(1+2a^2+2b^2)}{(3-2a)(2+a)(3-2b)(2+b)}\\&\ge 0\end{aligned}$$
from which the desired inequality follows.

Remark: One may also prove the inequality
$$\sum_{cyc}\frac{1}{3b^2+3c^2+a^2}\ge\sum_{cyc}\frac{1}{a^2+2}=\sum_{cyc}\frac{1}{3a^2+2b^2+2c^2}$$
by without loss of generality assuming $a\ge b\ge c$, appealing to the majorization relation
$$(3a^2+3b^2+c^2,3a^2+b^2+3c^2,a^2+3b^2+3c^2)\succ(3a^2+2b^2+2c^2,2a^2+3b^2+2c^2,2a^2+2b^2+3c^2)$$
and the convexity of the function $f(x)=\frac{1}{x}$, and applying Karamata's inequality.

2. Let γ be the incircle of $\triangle ABC$ (i.e. the circle inscribed in $\triangle ABC$) and I be the center of γ. Let D, E and F be the feet of the perpendiculars from I to BC, CA and AB respectively. Let D' be the point on γ such that DD' is a diameter of γ. Suppose the tangent to γ through D intersects the line EF at P. Suppose the tangent to γ through D' intersects the line EF at Q. Prove that $\angle PIQ+\angle DAD'=180°$.

Solution

Let M be the point of intersection of AI and EF. Since $IM \cdot IA = IE^2 = IF^2 = ID^2 = ID'^2$, it follows that $\triangle IDM \sim \triangle IAD$ and $\triangle ID'M \sim \triangle IAD'$, and hence $\angle DAD' = \angle DAI + \angle D'AI = \angle IDM + \angle ID'M$. Since $DIMP$ and $D'QIM$ are cyclic quadrilaterals, we have $\angle IDM = \angle IPM$ and $\angle ID'M = \angle IQM$. Therefore, $\angle DAD' = \angle IPM + \angle IQM = 180° - \angle PIQ$ as desired.

3. A *graph* consists of a set of vertices, some of which are connected by (undirected) edges. A *star* of a graph is a set of edges with a common endpoint. A *matching* of a graph is a set of edges such that no two have a common endpoint. Show that if the number of edges of a graph G is larger than $2(k-1)^2$, then G contains a matching of size k or a star of size k.

   Solution

   First, assume that there is no star of size k, as if there is, then we are trivially done.

   Then, for any arbitrary edge AB, we know that both vertices A and B have at most $k-1$ edges, or $k-2$ other edges. Hence, there are at most $2k-4$ adjacent edges. Now, any edge that is not AB and is not one of the at most $2k-4$ adjacent edges has no endpoint in common with AB, so they satisfy the conditions of a matching.

   Then, consider the following operation - choose any arbitrary edge A_1B_1, and remove it and the at most $2k-4$ adjacent edges from the graph.

   Repeat this operation on the graph, that is, on the new graph with at most $2k-3$ less edges, choose another arbitrary edge A_2B_2 and remove it and its adjacent edges from the graph. Then, any edge left in the graph satisfies the conditions of a matching with both A_1B_1 and A_2B_2.

   Repeat this operation $k-1$ times, as $|E(G)| > 2(k-1)^2 > (k-1)(2k-3)$. Then, A_iB_i for $i = 1, \cdots, k-1$ satisfy the conditions of a matching, as does any edge still left in the graph, which much exist as we have removed at most $(k-1)(2k-3)$ edges. Therefore, we found k edges that satisfy the conditions of a matching, so a matching exists.