1. Prove that
\[
\frac{1}{a^2 + 2} + \frac{1}{b^2 + 2} + \frac{1}{c^2 + 2} \leq \frac{1}{6ab + c^2} + \frac{1}{6bc + a^2} + \frac{1}{6ca + b^2}
\]
for any positive real numbers \(a, b\) and \(c\) satisfying \(a^2 + b^2 + c^2 = 1\).

**Solution**

By the AM-GM inequality, \(3\frac{a^2}{3} + 3\frac{b^2}{3} \geq 6ab\), \(3\frac{b^2}{3} + 3\frac{c^2}{3} \geq 6bc\) and \(3\frac{c^2}{3} + 3\frac{a^2}{3} \geq 6ca\). Hence

\[
\frac{1}{3a^2 + 3b^2 + c^2} + \frac{1}{3b^2 + 3c^2 + a^2} + \frac{1}{3c^2 + 3a^2 + b^2} \leq \frac{1}{6ab + c^2} + \frac{1}{6bc + a^2} + \frac{1}{6ca + b^2}.
\]

Now

\[
\sum_{cyc} \frac{1}{3b^2 + 3c^2 + a^2} - \sum_{cyc} \frac{1}{a^2 + 2} = \sum_{cyc} \left( \frac{1}{3 - 2a^2} - \frac{1}{a^2 + 2} \right)
\]

\[
= \sum_{cyc} \frac{3a^2 - 1}{(3 - 2a^2)(2 + a^2)}
\]

\[
= \sum_{cyc} \left( \frac{a^2 - b^2}{(3 - 2a^2)(2 + a^2)} - \frac{c^2 - a^2}{(3 - 2a^2)(2 + a^2)} \right)
\]

\[
= \sum_{cyc} \left( \frac{a^2 - b^2}{(3 - 2a^2)(2 + a^2)} - \frac{a^2 - b^2}{(3 - 2b^2)(2 + b^2)} \right)
\]

\[
= \sum_{cyc} \frac{(a - b)^2(1 + 2a^2 + 2b^2)}{(3 - 2a)(2 + a)(3 - 2b)(2 + b)}
\]

\[
\geq 0
\]

from which the desired inequality follows.

**Remark:** One may also prove the inequality by without loss of generality assuming \(a \geq b \geq c\), appealing to the majorization relation

\(3a^2 + 3b^2 + c^2, 3a^2 + b^2 + 3c^2, a^2 + 3b^2 + 3c^2 \succ (3a^2 + 2b^2 + 2c^2, 2a^2 + 3b^2 + 2c^2, 2a^2 + 2b^2 + 3c^2)\)

and the convexity of the function \(f(x) = \frac{1}{x}\), and applying Karamata’s inequality.

2. Let \(\gamma\) be the incircle of \(\triangle ABC\) (i.e. the circle inscribed in \(\triangle ABC\)) and \(I\) be the center of \(\gamma\). Let \(D, E\) and \(F\) be the feet of the perpendiculars from \(I\) to \(BC, CA\) and \(AB\) respectively. Let \(D'\) be the point on \(\gamma\) such that \(DD'\) is a diameter of \(\gamma\). Suppose the tangent to \(\gamma\) through \(D\) intersects the line \(EF\) at \(P\). Suppose the tangent to \(\gamma\) through \(D'\) intersects the line \(EF\) at \(Q\). Prove that \(\angle PIQ + \angle DAD' = 180^\circ\).
Solution
Let $M$ be the point of intersection of $AI$ and $EF$. Since $IM \cdot IA = IE^2 = IF^2 = ID^2 = ID'^2$, it follows that $\triangle IDM \sim \triangle IAD$ and $\triangle ID'M \sim \triangle IAD'$, and hence $\angle DAD' = \angle DAI + \angle D'AI = \angle IDM + \angle ID'M$. Since $DIIP$ and $D'QIM$ are cyclic quadrilaterals, we have $\angle IDM = \angle IPM$ and $\angle ID'M = \angle IQM$. Therefore, $\angle DAD' = \angle IPM + \angle IQM = 180^\circ - \angle PIQ$ as desired.

3. A graph consists of a set of vertices, some of which are connected by (undirected) edges. A star of a graph is a set of edges with a common endpoint. A matching of a graph is a set of edges such that no two have a common endpoint. Show that if the number of edges of a graph $G$ is larger than $2(k-1)^2$, then $G$ contains a matching of size $k$ or a star of size $k$.

Solution
First, assume that there is no star of size $k$, as if there is, then we are trivially done.

Then, for any arbitrary edge $AB$, we know that both vertices $A$ and $B$ have at most $k-1$ edges, or $k-2$ other edges. Hence, there are at most $2k-4$ adjacent edges. Now, any edge that is not $AB$ and is not one of the at most $2k-4$ adjacent edges has no endpoint in common with $AB$, so they satisfy the conditions of a matching.

Then, consider the following operation - choose any arbitrary edge $A_1B_1$, and remove it and the at most $2k-4$ adjacent edges from the graph.

Repeat this operation on the graph, that is, on the new graph with at most $2k-3$ less edges, choose another arbitrary edge $A_2B_2$ and remove it and its adjacent edges from the graph. Then, any edge left in the graph satisfies the conditions of a matching with both $A_1B_1$ and $A_2B_2$.

Repeat this operation $k-1$ times, as $|E(G)| > 2(k-1)^2 > (k-1)(2k-3)$. Then, $A_iB_i$ for $i = 1, \cdots, k-1$ satisfy the conditions of a matching, as does any edge still left in the graph, which much exist as we have removed at most $(k-1)(2k-3)$ edges. Therefore, we found $k$ edges that satisfy the conditions of a matching, so a matching exists.