1. Let \( a_1 = 2013 \) and \( a_{n+1} = 2013^{a_n} \) for all positive integers \( n \). Let \( b_1 = 1 \) and \( b_{n+1} = 2013^{2012b_n} \) for all positive integers \( n \). Prove that \( a_n > b_n \) for all positive integers \( n \).

Solution

We claim that \( a_n \geq 2013b_n \) for all positive integers \( n \). This is clearly true for \( n = 1 \). If \( a_k \geq 2013b_k \) for some positive integer \( k \), then 

\[
\begin{align*}
    a_{k+1} &= 2013^{a_k} \\
    &\geq 2013^{2013b_k} = 2013^{b_k} \cdot 2013^{2012b_k} \geq 2013b_{k+1}.
\end{align*}
\]

2. Find all pairs of positive integers \((a, b)\) such that 

\[
\frac{a^3 + 4b}{a + 2b^2 + 2a^2b}
\]

is a positive integer.

Solution

When \( a = 1 \), we have \( b = 1 \) as a unique solution. When \( b = 1 \), we have

\[
\frac{a^3 + 4}{2a^2 + a + 2} \in \mathbb{Z}
\]

and we can easily check that \( a = 1, 2 \) are the only solutions. Assume from this point that \( a, b \geq 2 \).

If \( 2 \leq a \leq b \), we must have \( a^3 \leq a^2b \) and \( 4b \leq 2b^2 \) so the value is smaller than 1, thus cannot be an integer. So assume \( a > b \). We have that

\[
\begin{align*}
    \frac{a^3 + 4b}{a + 2b^2 + 2a^2b} &\in \mathbb{Z} \\
    \Rightarrow \frac{2b(a^3 + 4b)}{a + 2b^2 + 2a^2b} &\in \mathbb{Z} \\
    \Rightarrow \frac{2b(a^3 + 4b)}{a + 2b^2 + 2a^2b} - a(a + 2b^2 + 2a^2b) = \frac{8b^2 - a^2 - 2ab^2}{a + 2b^2 + 2a^2b} &\in \mathbb{Z} 
\end{align*}
\]

Now, for \( a \geq 4 \), we have that \( 8b^2 - 2ab^2 - a^2 < 0 \), thus

\[
\frac{a^2 + 2ab^2 - 8b^2}{a + 2b^2 + 2a^2b} \in \mathbb{N}
\]

However, we have \((a^2 + 2ab^2) - (b^2 + 2a^2b) = (a - b)(a + b - 2ab) < 0\), so numerator is smaller than denominator. Contradiction.

Thus we must have \( 4 > a > b \geq 2 \): the only possible case is \( a = 3, b = 2 \), and in that case the value becomes \( \frac{35}{3+12+36} \) which is not an integer.

Thus the only solutions are \((a, b) = (2, 1), (1, 1)\).
3. Find the smallest positive integer $n$ with the following property: for every sequence of positive integers $a_1, a_2, ..., a_n$ with $a_1 + a_2 + ... + a_n = 2013$, there exist some (possibly one) consecutive term(s) in the sequence that add up to 70.

**Solution**

We claim that $n \geq 1034$. If $n = 1033$, then consider the 70-term subsequence

$$1, 1, ..., 1, 71$$

with 69 1’s. The sum of the terms in this subsequence is 140. Now consider the sequence obtained by concatenating 14 copies of this subsequence and then 53 1’s. The sum of the terms in this sequence is $14 \times 140 + 53 = 2013$ and the sequence contains no consecutive terms that add up to 70, a contradiction. If $n < 1033$, we can also construct a counterexample by combining some terms in the above sequence to form a new sequence with fewer terms. Thus $n \geq 1034$.

We now show that the minimum value of $n$ is indeed 1034. Put the positive integers from 1 to 2013 into the following sets:

\[
\begin{align*}
\{1, 71\}, & \quad \{2, 72\}, \quad \ldots \quad \{70, 140\} \\
\{141, 211\}, & \quad \{142, 212\}, \quad \ldots \quad \{210, 280\} \\
\ldots & \\
\{1821, 1891\}, & \quad \{1822, 1892\}, \quad \ldots \quad \{1890, 1960\} \\
\{1961\}, & \quad \{1962\}, \quad \ldots \quad \{2013\}
\end{align*}
\]

Define $S_k = a_1 + a_2 + ... + a_k$. By the pigeonhole principle, there must be some $S_i$ and $S_j$ ($1 \leq i < j \leq 1034$) that belong to the same pair above. Then $a_{i+1} + a_{i+2} + ... + a_j = S_j - S_i = 70$. 

2