Individual Finals A Solutions

1. Let p be a prime number greater than 5. Prove that there exists a positive integer n such that p divides $20^n + 15^n - 12^n$.

   Solution:

   I claim that $n = p - 3$ works. Using the cool "Pythagorean triple" $\frac{1}{20^2} + \frac{1}{15^2} = \frac{1}{12^2}$, we have $20^{p-3} + 15^{p-3} - 12^{p-3} = \frac{20^{p-1}}{20^2} + \frac{15^{p-1}}{15^2} - \frac{12^{p-1}}{12^2} = \frac{20^{p-1}-1}{20^2} + \frac{15^{p-1}-1}{15^2} - \frac{12^{p-1}-1}{12^2} = \frac{9(20^{p-1}-1)+16(15^{p-1}-1)-25(12^{p-1}-1)}{3600}$.

   Since we know that this fraction is an integer, to show that it is divisible by p it suffices to check that the numerator is divisible by p and the denominator is not. Since $p > 5$, p is relatively prime to 12, 15, and 20, so by Fermat's little theorem we see that p divides the numerator. Also since $p > 5$ and $3600 = 2^4 \cdot 3^2 \cdot 5^2$, we see that p does not divide the denominator. We conclude that p divides $20^{p-3} + 15^{p-3} - 12^{p-3}$.

   The intuition for this solution is quite simple: since $f(n) = 20^n + 15^n - 12^n$ equals 0 for $n = -2$ and $f(n) \bmod p$ is periodic with some period dividing $p - 1$ for $n \geq 0$ (by Fermat's little theorem), we ought to have $f(-2 + (p-1)) \equiv 0 \pmod{p}$.

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2. Let a, b, c be real numbers such that $a + b + c = abc$. Prove that $\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} \geq \frac{3}{4}$.

   Solution:

   As the condition and the inequality are invariant under the transformation $(a, b, c) \to (-a, -b, -c)$, we may assume that at most one of a, b, c is negative, so WLOG let $a, b \geq 0$. Let $A, B \in \left[0, \frac{\pi}{2}\right)$ be such that $\tan A = a$ and $\tan B = b$, and let $C = \pi - A - B$. Then, $c = \frac{a+b}{ab-1} = -\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan(A + B) = \tan C$.

   We have $\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = \frac{1}{\tan^2 A+1} + \frac{1}{\tan^2 B+1} + \frac{1}{\tan^2 C+1} = \cos^2 A + \cos^2 B + \cos^2 C = \cos^2 A + \cos^2 B + (\sin A \sin B - \cos A \cos B)^2 = \cos^2 A + \cos^2 B + (1 - \cos^2 A)(1 - \cos^2 B) - 2 \sin A \sin B \cos A \cos B + \cos^2 A \cos^2 B = 1 - 2 \cos A \cos B(\sin A \sin B - \cos A \cos B) = 1 - 2 \cos A \cos B \cos C$.

   We now show that $\cos A \cos B \cos C \leq \frac{1}{8}$. Since $A, B \in \left[0, \frac{\pi}{2}\right)$, $\cos A \geq 0$ and $\cos B \geq 0$, so if $\cos C < 0$ then $\cos A \cos B \cos C \leq 0 \leq \frac{1}{8}$. Otherwise we may assume $\cos C \geq 0$, so by AM-GM we have $\cos A \cos B \cos C \leq \left(\frac{\cos A+\cos B+\cos C}{3}\right)^3$. Finally, since $A, B, C \in [0, \pi]$ and $\cos x$ is concave on this interval, we have by Jensen's that $\frac{\cos A+\cos B+\cos C}{3} \leq \cos\left(\frac{A+B+C}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. Putting this together gives $\cos A \cos B \cos C \leq \frac{1}{8}$, so $\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = 1 - 2 \cos A \cos B \cos C \geq \frac{3}{4}$.

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3. Let ABC be a triangle with incenter I, and let D be the foot of the angle bisector from A to BC. Let Γ be the circumcircle of triangle BIC, and let PQ be a chord of Γ passing through D. Prove that AD bisects $\angle PAQ$.

Solution:

If $\overline{PQ} = \overline{BC}$ the result is trivial, so we may assume otherwise.

Let $m\angle ABC = b$, $m\angle BCA = c$, and $m\angle CAB = a$. Also, let T be the center of Γ and let Ω be the circumcircle of triangle ABC. We first claim that T lies on Ω. As $m\angle ICB = \frac{c}{2}$, we have $m\angle ITB = c$. Similarly, $m\angle ITC = b$. Thus $m\angle BAC + m\angle BTC = a + (b + c) = \pi$, so quadrilateral $ACTB$ is cyclic. As $\overline{BT}$ and $\overline{CT}$ are chords of Ω with equal length, we must have $m\angle BAT = m\angle CAT$, so T lies on line $\overleftrightarrow{AID}$.

We now wish to show that quadrilateral $AQTP$ is cyclic. Let Λ be the circumcircle of triangle APQ. Since one of P, Q lies inside Ω and the other lies outside Ω, Λ and Ω must intersect in exactly two points, and we let the point of intersection which is not A be called T'. As Λ and Ω have radical axis $\overline{AT'}$, Λ and Γ have radical axis $\overline{PQ}$, and Ω and Γ have radical axis $\overline{BC}$, it follows by the radical axis theorem that these three line segments must be concurrent. As $\overline{BC}$ and $\overline{PQ}$ intersect at point D, we see that $\overline{AT'}$ must pass through D, so T' lies both on $\overleftrightarrow{AD}$ and on Ω. $\overleftrightarrow{AD}$ and Ω intersect only at A and T, and as $T' \neq A$, it follows that $T' = T$. Thus $AQTP$ is cyclic.

As $\left|\overline{PT}\right| = \left|\overline{QT}\right|$, we see that $m\angle TPQ = m\angle TQP$. Since $AQTP$ is cyclic, we conclude that $m\angle DAP = m\angle TAP = m\angle TQP = m\angle TPQ = m\angle TAQ = m\angle DAQ$, and the result follows.

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