PUMaC 2012

Individual Finals B

1. Let q be a fixed odd prime. A prime p is said to be *orange* if for every integer a there exists an integer r such that $r^q \equiv a \pmod{p}$. Prove that there are infinitely many orange primes.

   Solution:

   We claim that a prime p is orange whenever $p \not\equiv 1 \pmod{q}$. Indeed, for such p, we know that $(p-1, q) = 1$, so there exist $m, n \in \mathbb{Z}$ such that $(p-1)m + qn = 1$. Thus, for every integer $a \not\equiv 0 \pmod{p}$, we have $a = a^{(p-1)m+qn} = \left(a^{p-1}\right)^m \cdot (a^n)^q \equiv (a^n)^q \pmod{p}$ (where the last step follows by Fermat's little theorem), so $r = a^n$ works. For $a \equiv 0 \pmod{p}$, we may take $r = 0$.

   It remains to show that there are infinitely many primes $p \not\equiv 1 \pmod{q}$. Assume on the contrary that there are only finitely many such primes, and let these primes be $p_1, p_2, \ldots, p_k$. Let $N = qp_1p_2 \cdots p_k - 1$. Then N is not divisible by any p_j, so all prime factors of N must be congruent to $1 \pmod{q}$, contradicting the fact that $N \equiv -1 \pmod{q}$.

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2. Let $O_1, O_2, \ldots, O_{2012}$ be 2012 circles in the plane such that no circle intersects or contains any other circle and no two circles have the same radius. For each $1 \le i < j \le 2012$, let $P_{i,j}$ denote the point of intersection of the two external tangent lines to O_i and O_j, and let T be the set of all $P_{i,j}$ (so $|T| = \binom{2012}{2} = 2023066$). Suppose there exists a subset $S \subset T$ with $|S| = 2021056$ such that all points in S lie on the same line. Prove that all points in T lie on the same line.

   Solution:

   For each O_i, let S_i be the sphere which has O_i as a great circle. Then $P_{i,j}$ is the apex of the cone tangent to S_i and S_j. As all three of these apexes must lie in both of the planes which are externally tangent to the three spheres, they must lie along the line in which these two planes intersect, so they must be collinear.

   Once we have this result, the problem reduces to a graph theory question. Let l be the line containing the points of S, and let G be a simple graph with vertices $v_1, v_2, \ldots, v_{2012}$ such that, for each $1 \le i < j \le 2012$, v_i is adjacent to v_j if and only if $P_{i,j}$ lies on l. From the problem statement, we are given that $|E(G)| \ge 2021056$. Also, our lemma says that for any three vertices a, b, c, if a is adjacent to b and b is adjacent to c, then a is adjacent to c.

   We first claim that G is connected. Suppose for the sake of contradiction that it is not. Then there exists some $1 \le k \le 2011$ such that $V(G)$ can be partitioned into two sets of sizes k and $2012 - k$ with no edges going between them. This implies that $|E(G)| \le \binom{k}{2} + \binom{2012-k}{2} = \frac{k(k-1)+(2012-k)(2011-k)}{2} = k^2 - 2012k + 2023066$. As this is a convex function of k, it achieves its maximum at one of the endpoints of the interval, so $|E(G)| \le 1^2 - 2012 + 2023066 = 2021055$ (it takes the same value for $k = 2011$). This contradicts $|E(G)| \ge 2021056$, so we conclude that G is connected.

   We now claim that G is complete. Since G is connected, for any two vertices a and b there exists a path from a to b, say $au_0u_1 \ldots u_nb$. Repeated application of the lemma gives that a

is adjacent to $u_1, u_2, \ldots, u_n$, and finally to b. We conclude that every pair of vertices of G is adjacent, which gives the desired result.

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3. Find, with proof, all pairs (x, y) of integers satisfying the equation $3x^2 + 4 = 2y^3$.

Solution:

This equation can be rewritten as $(2+x)^3 + (2-x)^3 = (2y)^3$. By Fermat's last theorem, at least one of $2+x$, $2-x$, and y must be zero. The first case gives the solution $(-2, 2)$, the second case gives the solution $(2, 2)$, and the third case gives no solutions.

Here is some motivation for this solution. We first note that x must be even, so let $x = 2k$ (for some $k \in \mathbb{Z}$). Plugging this in and simplifying the equation gives us $6k^2 + 2 = y^3$. Thus y must be even as well, and letting $y = 2l$ gives $3k^2 + 1 = 4l^3$. Now we see that k must be odd, so letting $k = 2m + 1$ gives $3m^2 + 3m + 1 = l^3$. The left-hand side is easily recognizable as $(m+1)^3 - m^3$, so we may rewrite this equation as $(m+1)^3 = l^3 + m^3$. Thus either $m = -1$, $m = 0$, or $l = 0$, and by considering these three cases we obtain the same two solutions.

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