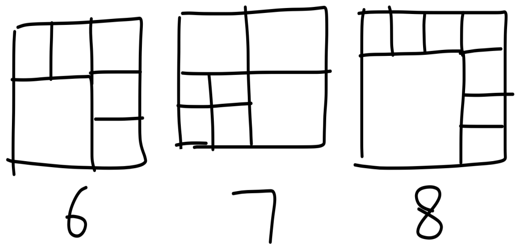
PUMaC 2014

Combinatorics A

1. [**3**] What is the largest n such that a square cannot be partitioned into n smaller, non-overlapping squares?

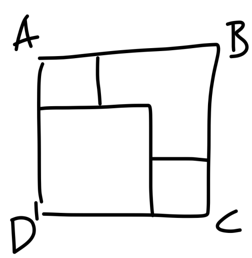
   Solution:

   We see that we can partion $n = 6, 7, 8$ as follow:



   For $n = 5$, we note that there needs to be a different square in every corner. (If one square is in two corners, it will be the size of the original square). Since the 5^{th} square can touch only 1 side without being in a corner, there can be at most 1 side with 3 squares touching it. Let this side be AB.

   Without loss of generality, let the corner square of D be larger than that of C. As can be seen, since there are only 2 squares on sides AD, DC, the configuration must be as such. However, we realise it is impossible for the corner square of B to be placed in any way such that the line BC is entirely contained.



   When the corner square of D and C are the same size, we see that their sides are half of the original square and for sides AD and BC to be covered completely, the corner squares of A, B also needs to be half of the original square. Hence there are only 4 squares.

   Thus we conclude that it is impossible for $n = \boxed{5}$.

2. [**3**] Assume you have a magical pizza in the shape of an infinite plane. You have a magical pizza cutter that can cut in the shape of an infinite line, but it can only be used 14 times. To

share with as many of your friends as possible, you cut the pizza in a way that maximizes the number of finite pieces (the infinite pieces have infinite mass, so you can't lift them up). How many finite pieces of pizza do you have?

Solution:

Consider a circle C that contains all points of intersection. Is is clear that all infinite pieces cannot be contained in C and there are no finite pieces outside C. Hence the number of infinite pieces is equal to the sectors outside of C. Since each line pass through C, there will be $2 \times 14 = 28$ infinite sectors outside C.

When there are already n lines, we see that the next line can intersect with at most n lines which will divide it into at most $n+1$ non-overlapping segments. Since each segment divide one existing area into two, there can be at most $n+1$ new areas. Hence with n lines there are $\frac{n(n+1)}{2}+1$ areas at most. (Since when $n=0$, there is already an area that is the entire plane. Hence with $n=14$, we have 106 areas, and $106-28=\boxed{78}$ of them are finite.

Next we show that this is possible. By drawing the lines all none parallel and no three lines concurrent, each line must intersect all others at distint points and thus each line will have $n+1$ distinct segments that makes $n+1$ new areas. Hence this construction will have 106 areas, 78 of them finite.

3. [**4**] You have three colors $\{red,\ blue,\ green\}$ with which you can color the faces of a regular octahedron (8 triangle sided polyhedron, which is two square based pyramids stuck together at their base), but you must do so in a way that avoids coloring adjacent pieces with the same color. How many different coloring schemes are possible? (Two coloring schemes are considered equivalent if one can be rotated to fit the other.)

Solution:

WLOG, let the (number of red pieces) $\geq$ (number of blue pieces) $\geq$ (number of green pieces). Then, we consider the subcases of when $r=4, b=4, g=0$, when $r=4, b=3, g=1$, when $r=4, b=2, g=2$, and when $r=3, b=3, g=2$.

For $r=4, b=4, g=0$, there is 1 possible coloring scheme. Accounting for color selection, we get $1 \cdot 3 = 3$.

For $r=4, b=3, g=1$, the piece opposite of the green piece must be red, because if it's blue, then it is impossible to place four red pieces between the green piece and the blue piece without having two adjacent red pieces. There is 1 possible coloring scheme that satisfies this. Accounting for color selection, we get $1 \cdot 6 = 6$.

For $r=4, b=2, g=2$, the piece opposite of each green or blue piece must be red, because otherwise, then it is impossible to place four red pieces between the green piece and its opposite piece without having two adjacent red pieces. There is 1 possible coloring scheme that satisfies this. Accounting for color selection, we get $1 \cdot 3 = 3$.

For $r = 3, b = 3, g = 2$, we consider several subcases.

Subcase 1: green is opposite green. Then there is 1 (3 when accounting for color selection) possible coloring scheme.

Subcase 2: green is not opposite green. Then there are no possible coloring schemes. To show this, WLOG fix a green piece and color the opposite piece blue (or red, doesn't matter). Then, one must fit three reds, two blues, and one green in between. The three reds must be in one of the two "rows" that comprise the middle portion of the visual presentation structure, because otherwise there would be two red pieces that are adjacent. It must be the top row, because otherwise the topmost blue would be adjacent to another blue inevitably. But that means inevitably green must be adjacent to green, so contradiction.

In total, we get $3 + 6 + 3 + 3 = \boxed{15}$.

4. [**4**] Amy has a 2×10 puzzle grid which she can use 1×1 and 1×2 (1 vertical, 2 horizontal) tiles to cover. How many ways can she exactly cover the grid without any tiles overlapping and without rotating the tiles?

Solution:

First we note that the two rows of 1×10 are independent of each other as there are no tiles that can overlap them both.

For a single row, the number of ways to tile a $1 \times n$ row, a_n is $a_{n-1} + a_{n-2}$ as if the last tile is a 1×1, there are a_{n-1} ways to tile the rest and if the last tile is 1×2, there are a_{n-2} ways to tile the rest. We see that since $a_1 = 1$ and $a_2 = 2$, we have $a_{10} = 89$.

Therefore there are $89^2 = \boxed{7921}$ ways to tile the entire 2×10 puzzle grid.

5. [**5**] What is the size of the largest subset S' of $S = \{2^x 3^y 5^z : 0 \leq x, y, z \leq 4\}$such that there are no distinct elements $p, q \in S'$ with $p|q$.

Solution: Let us find (x, y, z) such that $x + y + z = 6$. We see that when $x = 0, 1, 2, 3, 4$, there are $3, 4, 5, 4, 3$ sets respectively, which means there are 19 such (x, y, z). Since $x + y + z = 6$ in these sets, for $p|q$, we must have $p = q$.

Let us show that 20 is impossible. Firstly, note that (y, z) of all elements are distinct. Otherwise, we have $2^x 3^{y_0} 5^{z_0}$ and $2^{x'} 3^{y_0} 5^{z_0}$, and the smaller of the two will divide the larger. We shall show that if $(y, z) = \{(0, 0), (0, 1), (1, 0), (3, 4), (4, 3), (4, 4)\}$then there cannot be 20 distinct (y, z), and by pigeonhole principle, any set of 20 distinct (y, z) must contain one of the above.

If 2^x is in the subset, we can change 2^x to 2^4 and the subset will still work. In the new subset, we see that the only $(4, y, z)$ we can have is $y = z = 0$. Thus there must be 19 distinct (y, z) for $x = \{0, 1, 2, 3\}$ and hence by Pigeonhole Principle, we must have x_1, x_2 distinct such that there are 5 sets of (x_i, y, z). However, it is clear that for a fixed x_i, the only set of 5 (y, z) such that (x_i, y, z) are all in the subset is when $y + z = 5$. Thus it is impossible to find two such subsets. Contradiction.

If $2^x 3$ is in the subset, we can change $2^x 3$ to $2^4 3$ and the subset will still work. In the new subset, we see that the only $(4, y, z)$ we can have is $(4, 1, 0)$ and $(4, 0, 1)$. Hence there must be 18 distinct (y, z). Just as before, there must be x_1, x_2 distinct with 5 sets of (x_i, y, z). This leads to a similar contradiction.

The same aregument can be made for $2^x 3^4 5^4, 2^x 3^4 5^3$ and $2^x 3^3 5^4$ by replacing them with $3^4 5^4, 3^4 5^3$ and $3^3 5^4$ respectively.

Hence we see that $\boxed{19}$ is the maximum possible.

6. [**6**] Let $f(n)$ be the number of points of intersections of diagonals of a n-dimensional hypercube that is not the vertice of the cube. For example, $f(3) = 7$ because the intersection points of a cube's diagonals are at the centers of each face and the center of the cube. Find $f(5)$

Solutions:

Lemma 1: Each coordinate of a diagonal intersection is either a 0, a 1, or a $\frac{1}{2}$.

Proof: Each diagonal has a paramentric representation; each coordinate is either 0, 1, t, or 1-t, where $0 \leq t \leq 1$. At least two coordinates must be t or $1 - t$. Diagonals D_1 and D_2 (with parametric representations $D_1(t) = (x_1(t), x_2(t), ..., x_n(t))$ and $D_2(t) = (y_1(t), y_2(t), ..., y_n(t))$) intersect when t can be set so that the coordinates of D_1 and D_2 are the same. Since D_1 and D_2 are distinct, there must be at least one coordinate i for which their equations differ, and for which these equations are without loss of generality $x_i = 1 - t_1$ and $y_i = t_2$ (otherwise the diagonals could only meet at a vertex of the hypercube). There must be another coordinate j for which $x_j = t_1$ and $y_j = t_2$ (otherwise we could set $s = 1 - t_2$, describe D_2 in terms of s, and there would be no coordinate satisfying the conditions of i). Thus, D_1 and D_2 must meet at the point where $t_1 = t_2 = \frac{1}{2}$, so the intersection point's coordinates are each either 0, 1, or $\frac{1}{2}$, with at least two coordinates that are $\frac{1}{2}$.

Lemma 2: Let $x = (x_1, x_2, ..., x_n)$ be a point in n-space such that $x_i \in \{0, 1, \frac{1}{2}\}$ for each i, and such that $x_i = x_j = \frac{1}{2}$ for some distinct i and j. Then there exist two diagonals of the n-cube that intersect at x.

Proof: Without loss of generality, suppose $x_1 = x_2 = ... = x_k = \frac{1}{2}$, and $x_{k+1}, x_{k+2}, ..., x_n \in \{0, 1\}$. We define points $A = (0, 0, ..., 0, x_{k+1}, x_{k+2}, ..., x_n)$, $B = (1, 1, ..., 1, x_{k+1}, x_{k+2}, ..., x_n)$, $C = (1, 0, 0, ..., 0, x_{k+1}, x_{k+2}, ..., x_n)$, and $D = (0, 1, ..., 1, x_{k+1}, x_{k+2}, ..., x_n)$. Diagonals AB and CD intersect at x.

Thus, there is a one-to-one correspondence between intersection points and n-tuples consisting of 1s, 0s, and at least two $\frac{1}{2}$s. There are $3^n - n2^{n-1} - 2^n$ of these tuples (3^n total tuples consisting of 0s, 1s, and $\frac{1}{2}$s; $n2^{n-1}$ tuples with one $\frac{1}{2}$, and 2^n with only 0s and 1s), so $f(n) = 3^n - n2^{n-1} - 2^n$.

Hence $f(5) = 3^5 - 5 \times 2^4 - 2^5 = \boxed{131}$.

7. [**7**] Tom and Jerry are playing a game. In this game, they use pieces of paper with 2014 positions, in which some permutation of the numbers $1, 2, ...2014$ are to be written. (Each number will be written exactly once). Tom fills in a piece of paper first. How many pieces of paper must Jerry fill in to ensure that at least one of his pieces of paper will have a permutation that has the same number as Tom's in at least one positon?

   Solutions:

   Jerry writes 1 to 1008 for the first 1008 spots and for the 1008 different pieces of paper, he cycle through the numbers such that the i^{th} paper will have $i, i+1, ..., 1008, 1, 2, ..., i-1$for the first 1008 position. Hence if any of the numbers $\{1, 2, ..., 1008\}$ appears in the first 1008 position on Tom's paper, Jerry would have gotten it right. Otherwise, Tom would have written the first 1008 numbers in the 1009^{th} to 2014^{th} position, which is clearly impossible as there are more numbers than positions. Thus Jerry can get it right in $\boxed{1008}$ tries.

   With any 1007 pieces written, we shall show there exist a piece that doesn't coincide with any of the 1007 at any position. Let s_n be the set of 1007 numbers that occurs at the n^{th} position. We see that for the i^{th} position $1 \leq i \leq 1007$, we can put in some number x such that $x \notin s_i$ and x is distinct from the $i-1$ entries before which.

   We continue assigning numbers until it is impossible at the j^{th} position. Let the set of numbers in the first $j-1$ positions be s. Let S be the set of all 2008 numbers. We shall show it is possible to find $l < j$ such that we can switch the number at the l^{th} position to the j^{th} position and find another number for the l^{th} positon that avoids s_l. Assuming the contrary that no such l exist, let us consider s/s_j. Since $s \cup s_j = S$, we see that $|s/s_j| \geq 1007$. Also $S/s \in s_j$. For any positon l which has number in s/s_j, we need that $s \cup s_l = S$ which is equivalent to $S/s \in s_l$. Thus $S/s \in s_i$ for at least 1008 distinct i. this is clearly impossible. Hence we can always find a position l to swap. Continuing this algorithm, we see that we can always construct a piece of paper that is distinct at every position from any 1007pieces of paper that Jerry picks.

8. [**8**] There are 60 friends who want to visit each other's home during summer vacation. Everyday, they decide to either stay home or vist the home of everyone who stayed home that day. Find the minimum number of days required for everyone to have visted their friends' homes.

   Solution:

   In 8 days, let each student go out on 4 days. Since there are $\binom{8}{4} = 70$ ways to do so, we can let every student go out on distinct combination of 4 days. Thus we see that for any two students A and B, there is at least one day for which A goes out to vist while B stays home and vice versa. Hence everyone will have visited all others by $\boxed{8}$ days.

For 7 days, by Sperner's Theorem, we see that the maximum number of subsets of days such that there are no two subsets with one contained in the other, is $\binom{7}{3} = 35$. Hence 7 days is impossible. Since $60 > 35$, this can also be proven by grouping the combination of days into sets like $\{(1, 2, 3), (1, 2, 3, 4)\}$ where no two students can have their days going out from the same set, and applying pigeonhole principle after that.