Number Theory A

1. [**3**] Let $f(x) = x^3 + ax^2 + bx + c$ have solutions that are distinct negative integers. If $a+b+c = 2014$, find c.

   Solution:

   We have that $x_1 + x_2 + x_3 = -a$, $x_1x_2 + x_2x_3 + x_1x_3 = b$ and $x_1x_2x_3 = -c$. Thus, $(x_1 - 1)(x_2 - 1)(x_3 - 1) = -c - b - a - 1 = -2015$. $2015 = 5 \times 13 \times 31$, therefore the only way that x_i can be distinct negative integers is when $x_1 = -4$, $x_2 = -12$, $x_3 = -30$, which gives $c = \boxed{1440}$.

2. [**3**] What is the last digit of $17^{17^{17^{17}}}$?

   Solution:

   All we care is that $17^{17^{17}} \equiv 1^{17^{17}} \equiv 1 \bmod 4$. Therefore, since the last digit repeats every four products of 17s, the last digit must be $\boxed{7}$.

3. [**4**] Find the number of ending zeros of 2014! in base 9. Give your answer in base 9.

   Solution:

   The number of multiples of 3 in 2014! can be calculated as $\lfloor \frac{2014}{3} \rfloor + \lfloor \frac{2014}{9} \rfloor + \lfloor \frac{2014}{27} \rfloor + \lfloor \frac{2014}{81} \rfloor + \lfloor \frac{2014}{243} \rfloor + \lfloor \frac{2014}{729} \rfloor = 1002$. Therefore, there should be 501 ending zeros in base 9 for 2014!. Converting to base 9 gives $\boxed{616}$.

4. [**4**] Find the sum of all positive integer x such that $3 \times 2^x = n^2 - 1$ for some positive integer n.

   Solution:

   We see that $n^2 - 1 = (n+1)(n-1)$. Clearly, $n \neq 1$. Thus, we consider two cases:

   Case 1: $n + 1 = 3 \times 2^a$, $n - 1 = 2^b$.

   It is clear that $b \geq a$ as otherwise $n+1$ is at least 6 times of $n-1$ which is impossible. Hence $n + 1 - (n-1) = 2 = 2^a(3 - 2^{b-a})$. If $a = 1$, then $b - a = 1$ and $b = 2$. Hence $x = a + b = 3$ is a solution. When $a = 0$, $b - a = 0$ and hence $x = 0$.

   Case 2: $n+1 = 2^a$, $n-1 = 3 \times 2^b$. It is clear that $a > b$. Hence $n+1-(n-1) = 2 = 2^b(2^{a-b} - 3)$. If $b = 1$, then $a - b = 2$ and $a = 3$ Hence $x = a + b = 4$ is a solution. When $b = 2$, $a - b$ is not an integer.

   Thus the only solutions are $3, 4$ and hence answer is $\boxed{7}$.

5. [**5**] Find the number of pairs of integer solution (x, y) that satisfies the equation

   $$(x - y + 2)(x - y - 2) = -(x - 2)(y - 2)$$

   Solution:

Reorganize and we get $(x-y)^2+(x-2)^2+(y-2)^2=8$. The only possible scenarios are when two of the three terms on the left evaluate to 4, and the other one to zero. Every scenario gives two solutions: when $x-y=0$, $x=y=4$ or $x=y=0$; when $x-2=0$, $x=2, y=4$ or $x=2, y=0$; when $y-2=0$, $x=4, y=2$ or $x=0, y=2$.Hence there are $\boxed{6}$ pairs of solution.

6. [**6**] Given $S=\{2,5,8,11,14,17,20...\}$. Given that one can choose n different numbers from S, $\{A_1, A_2, ...A_n\}$, s.t. $\sum_{i=1}^{n}\dfrac{1}{A_i}=1$. Find the minimum possible value of n.

   Solution:

   It is clearly not possible that $n \leq 5$. We see that $\sum_{i=1}^{n}\dfrac{1}{A_i}=1$ can be rewritten as $\prod_{i=1}^{n}A_i = \sum_{i=1}^{n}\prod_{j\neq i}A_j$. Taking mod 3, we have $2^n \equiv n2^{n-1}$ which reduces to $n \equiv 2 \mod 3$. Hence the smallest possible $n=8$ and such an example for $\boxed{8}$ will be: $\{2,5,8,11,20,44,89,792\}$.

7. [**7**] Find the number of positive integers $n \leq 2014$ such that there exists integer x that satisfies the condition that $\dfrac{x+n}{x-n}$ is an odd perfect square.

   Solution:

   $\frac{x+n}{x-n}=k^2 \iff 1+\frac{2n}{x-n}=k^2 \Longrightarrow 2n=(x-n)(k^2-1)$

   By the problem condition, k^2-1 is even. Then, n is also even, since $k^2-1 \not\equiv 2 \pmod 4$. So, letting $k=2a-1$ and $n=2b$, we get $b=(x-2b)(a^2-a)$.

   It is sufficient to find all $b \leq 1007$ such that b is divisible by a^2-a. Note that $2^2-2=2$, so any even b works. However, since a^2-a is always even, no odd b works. So, there are $\boxed{503}$ integers n that satisfy the problem statement.

8. [**8**] Find all number sets (a,b,c,d) s.t. $1<a\leq b\leq c\leq d$, $a,b,c,d\in\mathbb{N}$, and $a^2+b+c+d$, $a+b^2+c+d$, $a+b+c^2+d$ and $a+b+c+d^2$ are all square numbers. Sum the value of d across all solution set(s).

   Solutions:

   We see that $(d+2)^2 > a+b+c+d^2 > d^2$ Hence $a+b+c+d^2=(d+1)^2=d^2+2d+1$. Hence we see that $a+b+c=2d+1$.

   We see that $2d+1\leq 3c$ and thus $d<\frac{3}{2}c$. Thus $c^2<a+b+c^2+d<c^2+\frac{7}{2}c<(c+2)^2$. We see that $a+b+c^2+d=(c+1)^2=c^2+2c+1$. Hence $a+b+d=2c+1$. With the previous equation, we see that $d=c$, and $a+b=d+1$.Thus $d+1\leq 2b$ and we have $d<2b$. Thus $b^2<a+b^2+c+d<b^2+b+4b$. Thus either $a+b^2+c+d=(b+1)^2$ or $a+b^2+c+d=(b+2)^2$.

   Case 1: $a+b^2+c+d=(b+1)^2$

   We have $a+c+d=2b+1$. Hence $b=c=d$. Thus $a+b^2+c+d=(b+1)^2-1+a$ is a square. Hence $a=1$ is the only solution, which is not in the range.

   Case 2: $a+b^2+c+d=(b+2)^2$

   We have that $a+c+d=4b+4$. We have from before $c=d$ and $a+b+c=2d+1$. Hence we can reduce the equations to $a+2c=4b+4$ and $a+b=c+1$. Hence $a+2(a+b-1)=4b+4$, which can be rewritten as $\frac{3}{2}a - 3 = b$, and $c = \frac{5}{2}a - 4$. Since $(a+1)^2 < a^2 + b + c + d = a^2 + \frac{3}{2}a - 3 + 5a - 8 < (a+4)^2$, we see that $a^2 + b + c + d = (a+2)^2$ or $a^2 + b + c + d = (a+3)^2$.

Case 2a: $a^2 + b + c + d = (a+2)^2$

Thus $b + c + d = 4a + 4$ which gives $a = b$. Hence $b + c + d = 4a + 4$ and $a + b + c = 2d + 1$ becomes $2c = 3a + 4$ and $2a = c + 1$. Solving, we have $2(2a-1) = 4a - 2 = 3a + 4$ which gives $a = 6$ and $d = 11$.

Case 2b: $a^2 + b + c + d = (a+3)^2$

Thus $b + 2c = 6a + 9$. We have $a + 2c = 4b + 4$. Thus we see that $5b + 4 = 7a + 9$. Putting this into $b + c + d = 6a + 9$ and $a + b + c = 2d + 1$ we have $2c = 6a + 8 - \frac{7a}{5}$ and $a + \frac{7a}{5} + 1 = c + 1$. Solving, we have that $a = 40, c = 96$ and hence $b = 57$ and $d = 96$.

Thus sum across all solution for d is $11 + 96 = \boxed{107}$