PUMaC 2014

Individual Finals A

1. Let γ be the incircle of $\triangle ABC$ (i.e. the circle inscribed in $\triangle ABC$) for which $AB+AC = 3BC$. Let the point where AC is tangent to γ be D. Let the incenter be I. Let the intersection of the circumcircle of $\triangle BCI$ with γ that is closer to B be P. Show that PID is colinear.

2. Given $a, b, c \in \mathbb{R}^+$, and that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{a^3+2} + \frac{1}{b^3+2} + \frac{1}{c^3+2} \geq 1$$

3. There are n coins lying in a circle. Each coin has two sides, $+$ and $-$. A *flop* means to flip every coin that has two different neighbors simultaneously, while leaving the others alone. For instance, $++-+$, after one *flop*, becomes $+---$.

   For n coins, let us define M to be a *perfect number* if for any initial arrangement of the coins, the arrangement of the coins after M *flops* is exactly the same as the initial one.

   (a) When $n = 1024$, find a perfect number M.

   (b) Find all n for which a perfect number M exist.