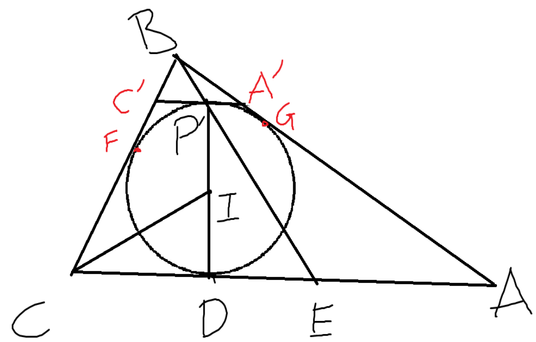
Individual Finals A

1. Let γ be the incircle of $\triangle ABC$ (i.e. the circle inscribed in $\triangle ABC$) for which $AB+AC = 3BC$. Let the point where AC is tangent to γ be D. Let the incenter be I. Let the intersection of the circumcircle of $\triangle BCI$ with γ that is closer to B be P. Show that PID is colinear.

   Solution:



   Let point P' be the diameter of γ passing through D. Hence we need to show that P and P' are the same point, meaning that $BP'IC$ is con-cyclic

   Since $AC+AB = 2AD+CD+BG = 2AD+CF+FB = 2AD+BC = 3BC$. Hence $AD = BC$.

   Let us construct $C'A'//CA$ passing through P'. Let F and G be the tangents of γ with the lines BC and BA respectively. We see that $BC'A' \simeq BCA$ and let the ratio of the sides $\frac{BC}{BC'} = k$. Extend BP' to meet AC at E.

   Hence we have $2AD = AG + DE + EA = AG + DE + kP'A' = AG + DE + kA'G = AG + DE + k(BG - BA') = AG + DE + kBG - BA = DE + kBG - BG = DE + (k-1)BG$

   Similarly, $2CE = CD + DE + CE = CF + DE + C'P'k = CF + DE + kC'F = CF + DE + k(BF - BC') = CF + DE + kBF - BC = DE + kBF - BF = DE + (k-1)BF$

   Hence we see that $CE = AD = BC$ and hence $\triangle CBE$ is isosceles $\angle CBE = \frac{180 - \angle C}{2}$. SInce $P'ID$ is the diameter, we see that $\angle IDC = 90^o$ and hence $\angle CID = \frac{180 - \angle C}{2}$ and therefore $BP'IC$ is concyclic, implying that $P' = P$ and PID is colinear.

2. Given $a, b, c \in \mathbb{R}^+$, and that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{a^3+2} + \frac{1}{b^3+2} + \frac{1}{c^3+2} \geq 1$$

   Solution:

We see that $\frac{1}{a^3+2} = \frac{1}{2}\left(1 - \frac{a^3}{a^3+2}\right)$. By $AM-GM$ on the denominator, we have $a^3+1+1 \geq 3a$. Hence $\frac{1}{2}\left(1 - \frac{a^3}{a^3+2}\right) \geq \frac{1}{2} - \frac{a^2}{3}$.

Therefore, we have $\frac{1}{a^3+2} + \frac{1}{b^3+2} + \frac{1}{c^3+2} \geq \frac{3}{2} - \frac{a^2+b^2+c^2}{3} = 1$

3. There are n coins lying in a circle. Each coin has two sides, $+$ and $-$. A *flop* means to flip every coin that has two different neighbors simultaneously, while leaving the others alone. For instance, $++-+$, after one *flop*, becomes $+---$.

   For n coins, let us define M to be a *perfect number* if for any initial arrangement of the coins, the arrangement of the coins after M *flops* is exactly the same as the initial one.

   (a) When $n = 1024$, find a perfect number M.

   (b) Find all n for which a perfect number M exist.

   Solution:

   (a) By part (b), since M exist, then any initial arrangement will cycle after x *flops* for some $x \leq 2^n$ Hence if we let $M = 2^n!$, $x|M$ will necessarily be true and hence at M *flops*, it will return to the initial position for the $\frac{M}{x}$ time. Thus $(2^{1024})!$ is a perfect number.

   (b) We see that for $3|n$, there can be no perfect number as $++-++-...++-$ after one *flop* will become $---...---$ and it is clear that this will remain this way for any future *flops* and therefore can never return to its initial arrangement.

   For $3 \nmid n$, let us consider the following: We can replace the $+$ with 1 and $-$ with -1. Let $a_{k,x}$ denote the value of the k^{th} coin after x *flops*. We see that with each *flop*, $a_{k,x} = a_{k-1,x-1}a_{k,x-1}a_{k+1,x-1}$. Note that $a_{k,x}^2 = 1$

   We see that for $n = 3m+2$, we can write

$$a_{k,x} = a_{k,x+1}a_{k+2,x+1}a_{k+3,x+1}a_{k+5,x+1}a_{k+6,x+1}...a_{k-3,x+1}a_{k-2,x+1}$$

   and expanding, we can verify this to be true as

$$a_{k,x} = (a_{k-1,x}a_{k,x}a_{k+1,x})(a_{k+1,x}a_{k+2,x}a_{k+3,x})(a_{k+2,x}a_{k+3,x}a_{k+4,x})...(a_{k-4,x}a_{k-3,x}a_{k-2,x})(a_{k-3,x}a_{k-2,x}a_{k-1,x})$$

   which we can group into

$$a_{k,x} = a_{k-1,x}^2 a_{k,x} a_{k+1,x}^2 a_{k+2,x}^2 + ...a_{k-3,x}^2 a_{k-2,x}^2$$

   which is clearly true.

   Similarly, we see that for $n = 3m+1$, we can write

$$a_{k,x} = a_{k,x+1}a_{k+1,x+1}a_{k+3,x+1}a_{k+4,x+1}a_{k+6,x+1}...a_{k-3,x+1}a_{k-1,x+1}$$

and expanding, we can verify this to be true as

$$a_{k,x} = (a_{k-1,x}a_{k,x}a_{k+1,x})(a_{k,x}a_{k+1,x}a_{k+2,x})(a_{k+2,x}a_{k+3,x}a_{k+4,x})...(a_{k-4,x}a_{k-3,x}a_{k-2,x})(a_{k-2,x}a_{k-1,x}a_{k,x})$$

which we can group into

$$a_{k,x} = a^2_{k-1,x}a^3_{k,x}a^2_{k+1,x}a^2_{k+2,x} + ...a^2_{k-3,x}a^2_{k-2,x}$$

which is clearly true.

Therefore, we see that for $3 \nmid n$, position after x *flops* will determine uniquely the position after $x-1$ and after $x+1$ *flops*. Since there is a finite number of positions possible, it must eventually cycle after some time and since any position uniquely determine the one before and after, the cycle must include the intitial position.

Hence M exist only for all n where $3 \nmid n$.