Team Round

Your team receives up to 100 points total for the team round. To play this minigame for up to 10 bonus points, you must decide how to construct an optimal army with number of soldiers equal to the points you receive.

Construct an army of 100 soldiers with 5 flanks; thus your army is the union of battalions B_1, B_2, B_3, B_4, and B_5. You choose the size of each battalion such that $|B_1| + |B_2| + |B_3| + |B_4| + |B_5| = 100$. The size of each battalion must be integral and non-negative. Then, suppose you receive n points for the Team Round. We will then "supply" your army as follows: if $n > B_1$, we fill in battalion 1 so that it has $|B_1|$ soldiers; then repeat for the next battalion with $n - |B_1|$ soldiers. If at some point there are not enough soldiers to fill the battalion, the remainder will be put in that battalion and subsequent battalions will be empty. (Ex: suppose you tell us to form battalions of size $\{20, 30, 20, 20, 10\}$, and your team scores 73 points. Then your battalions will actually be $\{20, 30, 20, 3, 0\}$.)

Your team's army will then "fight" another's. The B_i of both teams will be compared with the other B_i, and the winner of the overall war is the army who wins the majority of battalion fights. The winner receives 1 victory point, and in case of ties, both teams receive $\frac{1}{2}$ victory points.

Every team's army will fight everyone else's and the team war score will be the sum of the victory points won from wars. The teams with ranking x where $7k < x \leq 7(k+1)$ will earn $10 - k$ bonus points.

For example: Team Princeton decides to allocate its army into battalions with size $|B_1|, |B_2|, |B_3|, |B_4|, |B_5| = 20, 20, 20, 20, 20$. Team MIT allocates its army into battalions with size $|B_1|, |B_2|, |B_3|, |B_4|, |B_5| = 10, 10, 10, 10, 60$. Now suppose Princeton scores 80 points on the Team Round, and MIT scores 90 points. Then after supplying, the armies will actually look like $\{20, 20, 20, 20, 0\}$ for Princeton and $\{10, 10, 10, 10, 50\}$ for MIT. Then note that in a war, Princeton beats MIT in the first four battalion battles while MIT only wins the last battalion battle; therefore Princeton wins the war, and Princeton would win 1 victory point.

1. [**4**] The evilest number 666^{666} has 1881 digits. Let a be the sum of digits of 666^{666} and let b be the sum of digits of a and let c be the sum of digits of b. Find c.

2. [**4**] Given a Pacman of radius 1, and mouth opening angle 90^o, what is the largest (circular) pellet it can eat? The pellet must lie entirely outside the yellow portion and entirely inside the circumcircle of the Pacman. Let the radius be equal to $a\sqrt{b}+c$ where b is square free. Find $a+b+c$.

3. [**4**] How many integer x are there such that $\dfrac{x^2-6}{x-6}$ is a positive integer?

4. [**5**] ABC is a right triangle with AC = 3, BC = 4, AB = 5. Squares are erected externally on the sides of the triangle. Evaluate the area of the hexagon PQRSTU.

5. [**5**] How many sets of positive integers (a, b, c) satisfies $a > b > c > 0$ and $a + b + c = 103$?

6. [**6**] Find the sum of positive integer solutions of x for $\dfrac{x^2}{1716-x} = p$, where p is a prime. (If there are no solution, answer 0.)

7. [**6**] Let us consider a function $f : N \to N$ for which $f(1) = 1$,$f(2n) = f(n)$ and $f(2n+1) = f(2n)+1$. Find the number of values at which the maximum value of $f(n)$ is attained for integer n satisfying $0 < n < 2014$.

8. [**7**] Let $n^2 - 6n + 1 = 0$. Find $n^6 + \dfrac{1}{n^6}$

9. [**7**] Find the largest p_n such that $p_n + \sqrt{p_{n-1} + \sqrt{p_{n-2} + \sqrt{... + \sqrt{p_1}}}} \leq 100$, where p_n denotes the n^{th} prime number.

10. [**7**] A gambler has \$25 and each turn, if the gambler has a positive amount of money, a fair coin is flipped and if it is heads, the gambler gains a dollar and if it is tails, the gambler loses a dollar. But, if the gambler has no money, he will automatically be given a dollar (which counts as a turn). What is the expected number of turns for the gambler to double his money?

11. [**8**] $\triangle ABC$ has $AB = 4$ and $AC = 6$. Let point D be on line AB so that A is between B and D. Let the angle bisector of $\angle BAC$ intersect line BC at E, and

let the angle bisector of DAC intersect line BC at F. Given that $AE = AF$, find the square of the circumcircle's radius' length.

12. [**9**] Let n be the number of possible ways to place six orange balls, six black balls, and six white balls in a circle (two placements are considered equivalent if one can be rotated to fit the other). What is the remainder when n is divided by 1000?

13. [**9**] There is a right triangle $\triangle ABC$, in which $\angle A$ is the right angle. On side AB, there are three points X, Y, and Z that satisfy $\angle ACX = \angle XCY = \angle YCZ = \angle ZCB$ and $BZ = 2AX$. The smallest angle of $\triangle ABC$ is $\frac{a}{b}$ degrees, where a, b are positive integers such that $GCD(a, b) = 1$. Find $a + b$.

14. [**9**] Define function $f_k(x)$ (where k is a positive integer) as follows:

$$f_k(x) = (\cos kx)(\cos x)^k + (\sin kx)(\sin x)^k - (\cos 2x)^k$$

Find the sum of all distinct value(s) of k such that $f_k(x)$ is a constant function.

15. [**10**] Jason has n coins, among which at most one of them is counterfeit. The counterfeit coin (if there is any) is either heavier or lighter than a real coin. Jason's grandfather also left him an old weighing balance, on which he can place any number of coins on either side and the balance will show which side is heavier. However, the old weighing balance is in fact really really old and can only be used 4 more times. What is the largest number n for which is it possible for Jason to find the counterfeit coin (if it exist)?