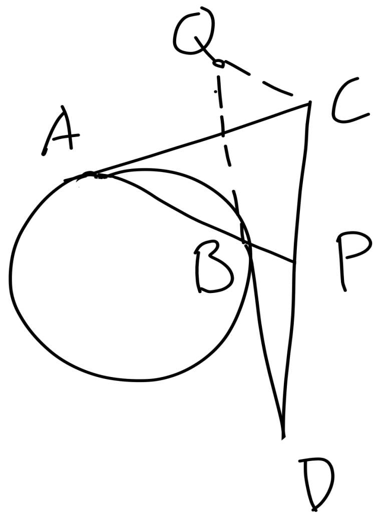
Individual Finals B

1. Let A, B be two points on circle γ. At point A and B we construct tangents to γ, AC and BD respectively such that the tangents are both in the clockwise direction. Let the intersection between AB and CD be P . If $AC = BD$, prove that P bisects the line CD.

   Solution:

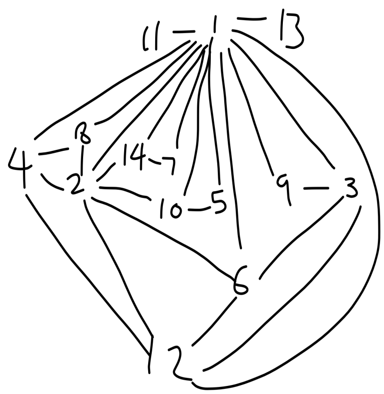
   

   Extend DB to Q as above such that $QB = BD = AC$. Since AC and BQ are equal tangents, by symmetry we see that $AB//QC$ and therefore $BP//QC$. Hence $\triangle QDC \simeq \triangle BDP$ and thus $\frac{CP}{PD} = \frac{QB}{BD} = 1$ and we are done.

2. Let $P_1, P_2, ..., P_n$ be points on the plane. There is an edge between distinct points P_x, P_y if and only if $x|y$. Find the largest n, such that the graph can be drawn with no crossing edges.

   Solution:

   We see that $n = 14$. The construction for $n = 14$ is as below:



For $n \geq 15$

We see that there are at least the following 3 paths between 2 and 3: $2-6-3$, $2-12-3$ and $2-10-5-15-3$. We see that these three paths are clearly distinct. Hence it will seperate the plane into 3 regions.

We see that no matter which region 1 falls into, the region is enclosed by 2 of the 3 paths and for any edge between 1 and the number on the 3^{rd} path, it must intersect the first two that encloses 1. Thus there will always be crossing edges.

3. Solve the following equation for $x, y \in \mathbb{N}$

$$x^3 + y^3 = 101xy + 101$$

Solution:

If $101|x$, we must also have $101|y$ and thus $x, y \geq 101$. Hence with out loss of generality, we can assume $x \geq y$ and thus we have $x^3 \geq 101xy$ and $y^3 > 101$ which means eqality cannot hold. Hence $101 \nmid x$ and $101 \nmid y$.

We see that $101|x^3 + y^3$ and hence $101|(x+y)(x^2 - xy + y^2)$. We wish to show that $101 \nmid (x^2 - xy + y^2)$. Assuming the contrary, let $101|(x^2 - xy + y^2)$. Hence we must have $101|(4x^2 - 4xy + y^2 + 3y^2) = (2x - y)^2 + 3y^2$. Hence we see that $-3y^2$ is a quadratic residue of 101 and hence, making use of the Legendre Symbol, we have $1 = \left(\frac{-3y^2}{101}\right) = \left(\frac{-1}{101}\right)\left(\frac{3}{101}\right)\left(\frac{y^2}{101}\right)$.

We see that $\left(\frac{-1}{101}\right) = (-1)^{(101-1)/2} = 1$ and since $101 \nmid y$, $\left(\frac{y^2}{101}\right) = 1$, we have $1 = \left(\frac{3}{101}\right) = \left(\frac{101}{3}\right)(-1)^{\frac{101-1}{2}\frac{3-1}{2}} = \left(\frac{2}{3}\right) = -1$which is contradictory and thus we must have $101 \nmid (x^2 - xy + y^2)$.

Hence $101|(x+y)$ and we have $x+y=101k$. Thus, we have $k(x^2+y^2)=(k+1)xy+1$ Since $k(x^2+y^2)\geq 2kxy$, we have $(k+1)xy+1\geq 2kxy$ and hence $xy+1\geq kxy$. Therefore $k=1$.

This leaves us with $x^2-xy+y^2=xy+1$. Hence $(x-y)^2=1$ and we see that the only solutions are $(x,y)=(51,50)$ or $(50,51)$.