1. [3] Triangle \(ABC\) has lengths \(AB = 20, AC = 14, BC = 22\). The median from \(B\) intersects \(AC\) at \(M\) and the angle bisector from \(C\) intersects \(AB\) at \(N\) and the median from \(B\) at \(P\). Let \(p = \frac{[AMPN]}{[ABC]}\) for positive integers \(p, q\) coprime. Note that \([ABC]\) denotes the area of triangle \(ABC\). Find \(p + q\).

Solution:
We have that \(\frac{[BMC]}{[ABC]} = \frac{MC}{AC} = \frac{1}{2}\) since \(BM\) is a median, and \(\frac{[BNC]}{[ABC]} = \frac{BN}{AB} = \frac{11}{18}\) from the angle bisector theorem. Now, we find the area of \(BPC\).

We see \(\frac{[BPC]}{[BMC]} = \frac{BP}{BM}\). Using mass points, \(A\) is assigned a mass of 11 meaning that the mass at \(M\) is 22 and the mass at \(B\) is 7. Therefore, \(\frac{BP}{BM} = \frac{22}{29}\). So \(\frac{[BNPMC]}{[ABC]} = 1 - \frac{11}{18} \cdot \frac{22}{29} = \frac{191}{261}\).

Then, \(\frac{[AMPN]}{[ABC]} = 1 - \frac{191}{261} = \frac{70}{261}\). Hence \(p + q = 331\).

2. [3] Consider the pyramid \(OABC\). Let the equilateral triangle \(ABC\) with side length 6 be the base. Also \(9 = OA = OB = OC\). Let \(M\) be the midpoint of \(AB\). Find the square of the distance from \(M\) to \(OC\).

Solution:
Let \(D\) be the center of the equilateral triangle. Take a slice of the pyramid that goes through the apex and \(MC\). Then we get a triangle with base \(3\sqrt{3}\) and \(OC = 9\). Dropping the perpendicular from \(O\) to \(D\), we have that \(OD^2 = 9^2 - (2\sqrt{3})^2 = OD = \sqrt{69}\). Now if we let the perpendicular from \(M\) to \(OC\) intersect \(OC\) at \(E\), we have that \(ME = \sqrt{23}\) \(\Rightarrow ME = \sqrt{23}\) \(\Rightarrow ME^2 = 23\).

3. [4] As given in figure (not drawn to proportion), in \(\triangle ABC\), \(E \in AC, D \in AB, P = BE \cap CD\). Given that \(S\triangle BPC = 12\), while the areas of \(\triangle BPD, \triangle CPE\) and quadrilateral \(AEPD\) are all the same, which is \(x\). Find the value of \(x\).

Solution:
Let \(S\triangle ADF = y\) and \(S\triangle AEF = z\). Hence \(x = y + z\). We see that \(\frac{z}{x} = \frac{\triangle AEF}{\triangle CEF} = \frac{\triangle AEB}{\triangle CEB} = \frac{2x}{x + 12}\) \(\text{and} \frac{y}{x} = \frac{\triangle ADF}{\triangle BDF} = \frac{\triangle ADC}{\triangle BDC} = \frac{2x}{x + 12}\). Solving, we see that \(1 = \frac{4x}{x + 12}\) and hence \(x = 4\).

4. [4] Let \(O\) be the circumcenter of triangle \(ABC\) with circumradius 15. Let \(G\) be the centroid of \(ABC\) and let \(M\) be the midpoint of \(BC\). If \(BC = 18\) and \(\angle MOA = 150^\circ\), find the area of \(OMG\).

Solution:
Since \(O\) is the circumcenter, we have that \(OM\) is perpendicular to \(BC\) and so \(OMB\) forms an equilateral triangle. Since \(OB = 15, BM = 9 \Rightarrow OM = 12\). Then we have that \(AO = \)
15 \Rightarrow [AOM] = \frac{1}{2}(12)(15) \sin 150 = 45. \text{ Then } G \text{ splits } AM \text{ into the ratio } 2:1 \text{ and so }

[OMG] = \frac{GM}{AM}[AOM] = \frac{1}{3} \times 45 = 15.

5. [5] Consider the cyclic quadrilateral with sides 1, 4, 8, 7 in that order. What is its circumdiameter? Let the answer be of the form a\sqrt{b} + c, for b square free. Find a + b + c

Solution:

Let the quadrilateral be denoted by \(ABCD\) with \(AB = 1, BC = 4, CD = 8, AD = 7\). We note that if we reflect point \(D\) across the perpendicular bisector of \(AC\) to \(D'\), we get the same circle. But then, we have \(AD' = 8, CD' = 7\) and so \(AB^2 + AD'^2 = BC^2 + D'C^2 = 65\) which is the circumdiameter squared. Hence the circumdiameter is \(\sqrt{65}\). Thus \(a + b + c = 66\).

6. [6] There is a point \(D\) on side \(AC\) of acute triangle \(\triangle ABC\). Let \(AM\) be the median drawn from \(A\) (so \(M\) is on \(BC\)) and \(CH\) be the altitude drawn from \(C\) (so \(H\) is on \(AB\)). Let \(I\) be the intersection of \(AM\) and \(CH\), and let \(K\) be the intersection of \(AM\) and line segment \(BD\). We know that \(AK = 8, BK = 8,\) and \(MK = 6\). Find the length of \(AI\).

Solution:

The line parallel to \(AM\) and passing through \(C\) meets line \(BD\) at some point (call this point \(E\)). Since \(M\) is on \(BC\), \(K\) is the midpoint of \(EB\). Let \(H'\) be the foot of perpendicular from \(K\) onto \(AB\). Since \(AK = 8 = KB\), we see that \(H'\) bisect \(AB\). Hence \(AE//KH'//CH\). Hence \(AICE\) is a parallelogram. Thus \(AI = CE = 2KM = 12\).

7. [7] Consider quadrilateral \(ABCD\). Given that \(\angle DAC = 70, \angle BAC = 40, \angle BDC = 20, \angle CBD = 35\). Let \(P\) be the intersection of \(AC\) and \(BD\). Find \(\angle BPC\).

Solution:

Take the circumcircle of triangle \(BCD\) with center \(O\). Then since \(CBD\) and \(COD\) both intersect arc \(CD\), we have that \(m\angle CBD = \frac{1}{2}m\angle COD \Rightarrow m\angle COD = 70\) and similary \(m\angle COB = 40\) and we can see that \(O = A\) and so \(A\) is the center of the circle. Suppose we extend \(AC\) to intersect the circle at the opposite side at \(E\), then we have that \(m\angle BPC = \frac{BC + DE}{2} = \frac{40 + 180 - 70}{2} = 75\).

8. [8] \(ABCD\) is a cyclic quadrilateral with circumcenter \(O\) and circumradius 7. \(AB\) intersects \(CD\) at \(E\), \(DA\) intersects \(CB\) at \(F\). \(OE = 13, OF = 14\). Let \(\cos \angle FOE = \frac{p}{q}\), with \(p, q\) coprime. Find \(p + q\).

Solution:

Since we’re given \(EO = 13\) and the radius is 7, by power of a point, the power of \(E\) with respect to \(\odot O\) is \(P(E) = (13 - 7)(13 + 7) = 120\). Similarly, the power of point \(F\) with respect to \(\odot O\) is \(P(F) = (14 - 7)(14 + 7) = 147\).

Construct point \(X\) on \(EF\) such that \(\angle CXE = \angle CDF\). Note that given \(ABCD\) is cyclic, this implies \(\angle CBA = 180 - \angle ADC = \angle CDF\). Thus note that

\[\angle EBC + \angle CXE = (180 - \angle CBA) + \angle CDF = 180.\]
Therefore quadrilateral $BEXC$ is cyclic; and similarly, quadrilateral $CXFD$ is cyclic. Note that the circle about $BEXC$ and $O$ have a common chord $BC$. Thus the power of point $F$ with respect to both circles is $FC \cdot FB$, and so the power of $F$ with respect to the circle about $BEXC$ is $P(F) = 147$. Similarly, the power of $E$ with respect to the circle about $CXFD$ is $P(E) = 120$.

Therefore, by power of a point, $(EX + XF)(XF) = 147 = P(F)$ and $(EX)(EX + XF) = 120 = P(E)$. Then by adding,

$$267 = 120 + 147 = (EX + XF)(EX) + (EX + XF)(XF) = (EX + XF)^2$$

$$\implies EF = \sqrt{267}.$$

Now by Law of Cosine on $\triangle EOF$,

$$EF^2 = EO^2 + OF^2 - 2 \cdot EO \cdot OF \cdot \cos(\angle EOF)$$

$$267 = 169 + 196 - 2 \cdot 13 \cdot 14 \cdot \cos(\angle EOF)$$

$$\implies \cos(\angle EOF) = \frac{-98}{-2 \cdot 13 \cdot 14} = \frac{7}{26}.$$

Hence $p + q = 33$. 