PUMaC 2014

Geometry B

1. [**3**] Triangle ABC has lengths $AB = 20, AC = 14, BC = 22$. The median from B intersects AC at M and the angle bisector from C intersects AB at N and the median from B at P. Let $\frac{p}{q} = \frac{[AMPN]}{[ABC]}$ for positive integers p, q coprime. Note that $[ABC]$ denotes the area of triangle ABC. Find $p + q$

   Solution:

   We have that $\frac{[BMC]}{[ABC]} = \frac{MC}{AC} = \frac{1}{2}$ since BM is a median, and $\frac{[BNC]}{[ABC]} = \frac{BN}{AB} = \frac{11}{18}$ from the angle bisector theorem. Now, we find the area of BPC.

   We see $\frac{[BPC]}{[BMC]} = \frac{BP}{BM}$. Using mass points, A is assigned a mass of 11 meaning that the mass at M is 22 and the mass at B is 7. Therefore, $\frac{BP}{BM} = \frac{22}{29}$. So $\frac{[BNPMC]}{[ABC]} = \frac{1}{2} + \frac{11}{18} - \frac{22}{29} \cdot \frac{1}{2} = \frac{191}{261}$. Then, $\frac{[AMPN]}{[ABC]} = 1 - \frac{191}{261} = \frac{70}{261}$. Hence $p + q = \boxed{331}$

2. [**3**] Consider the pyramid $OABC$. Let the equilateral triangle ABC with side length 6 be the base. Also $9 = OA = OB = OC$. Let M be the midpoint of AB. Find the square of the distance from M to OC.

   Solution:

   Let D be the center of the equilateral triangle. Take a slice of the pyramid that goes through the apex and MC. Then we get a triangle with base $3\sqrt{3}$ and $OC = 9$. Dropping the perpendicular from O to D, we have that $OD^2 = 9^2 - (2\sqrt{3})^2 \Rightarrow OD = \sqrt{69}$. Now if we let the perpendicular from M to OC intersect OC at E, we have that $ME \cdot OC = MC \cdot OD = 3\sqrt{3}\sqrt{69} = 9\sqrt{23} \Rightarrow ME = \sqrt{23} \Rightarrow ME^2 = \boxed{23}$.

3. [**4**] As given in figure (not drawn to proportion), in $\triangle ABC$, $E \in AC$, $D \in AB$, $P = BE \cap CD$ Given that $S\triangle BPC = 12$, while the areas of $\triangle BPD$, $\triangle CPE$ and quadrilateral $AEPD$ are all the same, which is x. Find the value of x.

   Solution:

   Let $S\triangle ADF = y$ and $S\triangle AEF = z$. Hence $x = y + z$. We see that $\frac{z}{x} = \frac{\triangle AEF}{\triangle CEF} = \frac{\triangle AEB}{\triangle CEB} = \frac{2x}{x+12}$ and $\frac{y}{x} = \frac{\triangle ADF}{\triangle BDF} = \frac{\triangle ADC}{\triangle BDC} = \frac{2x}{x+12}$. Solving, we see that $1 = \frac{4x}{x+12}$ and hence $x = \boxed{4}$.

4. [**4**] Let O be the circumcenter of triangle ABC with circumradius 15. Let G be the centroid of ABC and let M be the midpoint of BC. If $BC = 18$ and $\angle MOA = 150^o$, find the area of OMG.

   Solution:

   Since O is the circumcenter, we have that OM is perpendicular to BC and so OMB forms an equilateral triangle. Since $OB = 15, BM = 9 \Rightarrow OM + 12$. Then we have that $AO =$

$15 \Rightarrow [AOM] = \frac{1}{2}(12)(15)\sin 150 = 45$. Then G splits AM into the ratio $2:1$ and so $[OMG] = \frac{GM}{AM}[AOM] = \frac{1}{3}45 = \boxed{15}$.

5. [**5**] Consider the cyclic quadrilateral with sides 1, 4, 8, 7 in that order. What is its circumdiameter? Let the answer be of the form $a\sqrt{b}+c$, for b square free. Find $a+b+c$

   Solution:

   Let the quadrilaterial be denoted by $ABCD$ with $AB = 1, BC = 4, CD = 8, AD = 7$. We note that if we reflect point D across the perpendicular bisector of AC to D', we get the same circle. But then, we have $AD' = 8, CD' = 7$ and so $AB^2 + AD'^2 = BC^2 + D'C^2 = 65$ which is the circumdiameter squared. Hence the circumdiameter is $\sqrt{65}$. Thus $a+b+c = \boxed{66}$

6. [**6**] There is a point D on side AC of acute triangle $\triangle ABC$. Let AM be the median drawn from A (so M is on BC) and CH be the altitude drawn from C (so H is on AB). Let I be the intersection of AM and CH, and let K be the intersection of AM and line segment BD. We know that $AK = 8$, $BK = 8$, and $MK = 6$. Find the length of AI.

   Solution:

   The line parallel to AM and passing through C meets line BD at some point (call this point E). Since M is the midpoint of BC, K is the midpoint of EB . Let H' be the foot of perpendicular from K onto AB. Since $AK = 8 = KB$, we see that H' bisect AB. Hence $AE//KH'//CH$. Hence $AICE$ is a parallelogram. Thus $AI = CE = 2KM = \boxed{12}$.

7. [**7**] Consider quadrilateral $ABCD$. Given that $\angle DAC = 70$, $\angle BAC = 40$, $\angle BDC = 20$, $\angle CBD = 35$. Let P be the intersection of AC and BD. Find $\angle BPC$.

   Solution:

   Take the circumcircle of triangle BCD with center O. Then since CBD and COD both intersect arc CD, we have that $m\angle CBD = \frac{1}{2}m\angle COD \Rightarrow m\angle COD = 70$ and similary $m\angle COB = 40$ and we can see that $O = A$ and so A is the center of the circle.

   Suppose we extend AC to intersect the circle at the opposite side at E, then we have that $m\angle BPC = \frac{BC + DE}{2} = \frac{40 + 180 - 70}{2} = \boxed{75}$.

8. [**8**] $ABCD$ is a cyclic quadrilateral with circumcenter O and circumradius 7. AB intersects CD at E, DA intersects CB at F. $OE = 13$, $OF = 14$. Let $\cos\angle FOE = \frac{p}{q}$, with p, q coprime. Find $p+q$.

   Solution:

   Since we're given $EO = 13$ and the radius is 7, by power of a point, the power of E with respect to $\odot O$ is $P(E) = (13-7)(13+7) = 120$. Similarly, the power of point F with respect to $\odot O$ is $P(F) = (14-7)(14+7) = 147$.

   Construct point X on $\overline{EF}$ such that $\angle CXE = \angle CDF$. Note that given $ABCD$ is cyclic, this implies $\angle CBA = 180 - \angle ADC = \angle CDF$. Thus note that

$$\angle EBC + \angle CXE = (180 - \angle CBA) + \angle CDF = 180.$$

Therefore quadrilateral $BEXC$ is cyclic; and similarly, quadrilateral $CXFD$ is cyclic. Note that the circle about $BEXC$ and $\circ O$ have a common chord BC. Thus the power of point F with respect to both circles is $FC \cdot FB$, and so the power of F with respect to the circle about $BEXC$ is $P(F) = 147$. Similarly, the power of E with respect to the circle about $CXFD$ is $P(E) = 120$.

Therefore, by power of a point, $(EX + XF)(XF) = 147 = P(F)$ and $(EX)(EX + XF) = 120 = P(E)$. Then by adding,

$$\begin{aligned} 267 = 120 + 147 = (EX + XF)(EX) + (EX + XF)(XF) &= (EX + XF)^2 \\ \implies EF &= \sqrt{267}. \end{aligned}$$

Now by Law of Cosine on ΔEOF,

$$\begin{aligned} EF^2 &= EO^2 + OF^2 - 2 \cdot EO \cdot OF \cdot \cos(\angle EOF) \\ 267 &= 169 + 196 - 2 \cdot 13 \cdot 14 \cos(\angle EOF) \\ \implies \cos(\angle EOF) &= \frac{-98}{-2 \cdot 13 \cdot 14} = \frac{7}{26}. \end{aligned}$$

Hence $p + q = \boxed{33}$.