PUMaC 2014

Team Round

Your team receives up to 100 points total for the team round. To play this minigame for up to 10 bonus points, you must decide how to construct an optimal army with number of soldiers equal to the points you receive.

Construct an army of 100 soldiers with 5 flanks; thus your army is the union of battalions B_1, B_2, B_3, B_4, and B_5. You choose the size of each battalion such that $|B_1|+|B_2|+|B_3|+|B_4|+|B_5| = 100$. The size of each battalion must be integral and non-negative. Then, suppose you receive n points for the Team Round. We will then "supply" your army as follows: if $n > B_1$, we fill in battalion 1 so that it has $|B_1|$ soldiers; then repeat for the next battalion with $n-|B_1|$ soldiers. If at some point there are not enough soldiers to fill the battalion, the remainder will be put in that battalion and subsequent battalions will be empty. (Ex: suppose you tell us to form battalions of size $\{20, 30, 20, 20, 10\}$, and your team scores 73 points. Then your battalions will actually be $\{20, 30, 20, 3, 0\}$.)

Your team's army will then "fight" another's. The B_i of both teams will be compared with the other B_i, and the winner of the overall war is the army who wins the majority of battalion fights. The winner receives 1 victory point, and in case of ties, both teams receive $\frac{1}{2}$ victory points.

Every team's army will fight everyone else's and the team war score will be the sum of the victory points won from wars. The teams with ranking x where $7k < x \leq 7(k+1)$ will earn $10-k$ bonus points.

Ex: Team Princeton decides to allocate its army into battalions with size $|B_1|, |B_2|, |B_3|, |B_4|, |B_5| = 20, 20, 20, 20, 20$. Team MIT allocates its army into battalions with size $|B_1|, |B_2|, |B_3|, |B_4|, |B_5| = 10, 10, 10, 10, 60$. Now suppose Princeton scores 80 points on the Team Round, and MIT scores 90 points. Then after supplying, the armies will actually look like $\{20, 20, 20, 20, 0\}$ for Princeton and $\{10, 10, 10, 10, 50\}$ for MIT. Then note that in a war, Princeton beats MIT in the first four battalion battles while MIT only wins the last battalion battle; therefore Princeton wins the war, and Princeton would win 1 victory point.

1. [**4**] The evilest number 666^{666} has 1881 digits. Let a be the sum of digits of 666^{666} and let b be the sum of digits of a and let c be the sum of digits of b. Find c

   Solutions

   We see that $a < 1881 \times 9$ has atmost 5 digits. Hence $b < 5 \times 9$ has at most 2 digits. Since $9|666^{666}$, we see that $9|a$ and hence $9|b$. Thus $b \in \{9, 18, 27, 36\}$ and $c = \boxed{9}$.

2. [**4**] Given a Pacman of radius 1, and mouth opening angle 90^o, what is the largest (circular) pellet it can eat? The pellet must lie entirely outside the yellow portion and entirely inside the circumcircle of the Pacman. Let the radius be equal to $a\sqrt{b}+c$ where b is square free. Find $a+b+c$.

   Solution:

   Inscribe a smaller circle within the circle sector and draw the line from the center of the larger circle through the center of the inscribed circle to the edge of the circle. Then the length of this line segment is 1. Dropping perpendiculars from the center of the smaller circle, we get a square and thus the length is also $r + r\sqrt{2}$ where r is the radius of the smaller circle. So $r(1+\sqrt{2}) = 1 \Rightarrow r = \sqrt{2}-1 \Rightarrow a+b+c = \boxed{2}$.

3. [**4**] How many integer x are there such that $\frac{x^2-6}{x-6}$ is a positive integer?

   Solution:

   We see that $\frac{x^2-6}{x-6} = x+6+\frac{30}{x-6}$. Since there are 8 diviors of 30, we see that taking the divisors and adding 6 will be all the solutions for $x > 6$. For $x < 6$, the denominator is negative, hence we see that the numerator needs to be negative as well. Thus we check $x = 2, 1, 0, -1, -2$, and we see that $x = 1, 0$ are two additional solutions, giving us a total of $\boxed{8}$ solutions.

4. [**5**] ABC is a right triangle with AC = 3, BC = 4, AB = 5. Squares are erected externally on the sides of the triangle. Evaluate the area of the hexagon PQRSTU.

   Solution:

   The hexagon is made up of 3 triangles and 3 squares and ABC. The area of each of these triangles is the same as the area of ABC because the area of each of these triangles is $\frac{1}{2}ab\sin\theta$. The angle is supplementary to one of the angles in ABC and $\sin(180-\theta) = \sin\theta$ while the side lengths are the same. Hence the total area is: $3^2+4^2+5^2+4[ABC] = \boxed{74}$.

5. [**5**] How many sets of positive integers (a, b, c) satisfies $a > b > c > 0$ and $a+b+c = 103$?

   Solution:

   There are $\binom{103+2-3}{2} = 102 \times 101/2 = 51 \times 101$ ways of spliting 103 into 3 non empty piles. There are 3×51 ways to split such that 2 of the 3 piles are equal and no pile is empty, and it is impossible for all 3 piles to be of the same size.

   Hence there are $51 \times (101-3) = 51 \times 98$ ways of spliting 103 into distinct, non empty piles, and ordering them, we have $51 \times 98/6 = 17 \times 49 = \boxed{833}$ ways of doing so.

6. [**6**] Find the sum of positive integer solutions of x for $\frac{x^2}{1716-x} = p$, where p is a prime. (If there are no solution, answer 0.)

   Solution:

   We get that $x^2 + xp - 1716p = 0$, so $x_1x_2 = -1716p$ and $x_1 + x_2 = -p$. We know that $1716 = 2^2 \times 3 \times 11 \times 13$, and also x_1, x_2 are both divisible by p. Therefore, we try $p = 2, 3, 11, 13$ and get two solutions for p: when $p = 11$, $x_1 = 132$, $x_2 = -143$ and when $p = 13$, $x_1 = 143$, $x_2 = -156$. Thus, the sum of positive values of x are $132 + 143 = \boxed{275}$.

7. [**6**] Let us consider a function $f : N \to N$ for which $f(1) = 1, f(2n) = f(n)$ and $f(2n+1) = f(2n) + 1$. Find the number of values at which the maximum value of $f(n)$ is attained for integer n satisfying $0 < n < 2014$.

   Solution:

Function is the number of 1 in the binary representation of the number. $1023 = 1111111111$ (10 1s). For 11 1s, one will need 2047, which is greater than 2014. We see that $2015 = 11111011111$ and hence only $\boxed{5}$ values satisfies $f(n) = 10$.

8. [**7**] Let $n^2 - 6n + 1 = 0$. Find $n^6 + \frac{1}{n^6}$

Solution:

We see that we have $n + \frac{1}{n} = 6$. Hence $n^2 + \frac{1}{n^2} = 36 - 2 = 34$ and $n^4 + \frac{1}{n^4} = 34^2 - 2 = 1154$. Therefore we have that $(n^4 + \frac{1}{n^4})(n^2 + \frac{1}{n^2}) = 1154 \times 34$ Thus $n^6 + \frac{1}{n^6} = 1154 \times 34 - 34 = 1153 \times 34 = \boxed{39202}$.

9. [**7**] Find the largest p_n such that $p_n + \sqrt{p_{n-1} + \sqrt{p_{n-2} + \sqrt{... + \sqrt{p_1}}}} \leq 100$, where p_n denotes the n^{th} prime number.

Solution:

For a rough estimate of the answer, we would like to have $(100 - p_n)^2 - p_{n-1}$ to be no less than $100 - p_n$. Letting $p_n = p_{n-1}$ and we get that $100 - p_n$ has to be bigger than 44, so $p_n \leq 90$. The biggest prime that satisfies the condition is 89. We see that $p_n + \sqrt{p_{n-1} + \sqrt{p_{n-2} + \sqrt{... + \sqrt{p_1}}}} < 89 + \sqrt{89 + \sqrt{89 + \sqrt{89 + ...}}} = S$. Since $S = (S - 89)^2$, we see that $S = \frac{179 + sqrt(357)}{2} < \frac{179 + 20}{2} < 100$. The next prime after $\boxed{89}$ is 97 which obviously fails.

10. [**7**] A gambler has \$25 and each turn, if the gambler has a positive amount of money, a fair coin is flipped and if it is heads, the gambler gains a dollar and if it is tails, the gambler loses a dollar. But, if the gambler has no money, he will automatically be given a dollar (which counts as a turn). What is the expected number of turns for the gambler to double his money?

Solution:

If we let E_i be the expected number of turns for the gambler to reach \$50 when he currently has $\$i$, we have that $E_0 = 1 + E_1$ and $E_i = \frac{E_{i-1} + E_{i+1}}{2} + 1$ for $1 \leq i \leq 49$ with $E_{50} = 0$.

So we have $E_1 = \frac{E_2 + 1 + E_1}{2} + 1 \Rightarrow E_1 = E_2 + 3$ and $E_2 = \frac{E_3 + 3 + E_2}{2} + 1 \Rightarrow E_2 = E_3 + 5$. We claim that $E_i = E_{i+1} + 2 * i + 1$ and by using induction, we have that $E_{i+1} = \frac{E_{i+1} + 2 * i + 1 + E_{i+2}}{2} + 1 \Rightarrow E_{i+1} = E_{i+2} + 2 * i + 3$ and so the result holds.

Therefore, it becomes clear that $E_0 = E_i + i^2$ and so $E_{25} + 25^2 = E_{50} + 50^2 = 50^2 \Rightarrow E_{25} = \boxed{1875}$.

11. [**8**] $\triangle ABC$ has $AB = 4$ and $AC = 6$. Let point D be on line AB so that A is between B and D. Let the angle bisector of $\angle BAC$ intersect line BC at E, and let the angle bisector of DAC intersect line BC at F. Given that $AE = AF$, find the square of the circumcircle's radius' length.

Solution:

Let the length of the circumradius be R. Sine law states that $\frac{AB}{\sin C} = \frac{AC}{\sin B} = 2R$. So, $AB^2 + AC^2 = 4R^2(\sin^2 B + \sin^2 C)$.

Let H be the intersection point that results when drawing an altitude from A to line BC. Then we get:

$\angle BAH = \angle EAH \pm \angle BAE = 45° \pm \angle BAE$

WLOG let the above satisfy minus instead of plus (which means the analogous equation for $\angle CAH$ satisfies plus instead of minus). Then, we also get:

$\angle ACB = \angle AEB - \angle CAE = 45° - \angle CAE$

Since $\angle BAE = \angle CAE$, we see that $\angle BAH = \angle ACB$. So, $\triangle BAH$ is similar to $\triangle ACH$. A little more work will show that $sinB = cosC$, so $R^2 = \frac{16+36}{4} = \boxed{13}$.

12. [**9**] Let n be the number of possible ways to place six orange balls, six black balls, and six white balls in a circle (two placements are considered equivalent if one can be rotated to fit the other). What is the remainder when n is divided by 1000?

Solution:

We can partition the placements into four categories, based on the size of the smallest subpermutation that can be repeated to fill the whole circle: 3-3-3-3-3-3, 6-6-6, 9-9, and 18.

There are a total $3! = 6$ permutations of the form 3-3-3-3-3-3. Correcting for repetition, we get $\frac{6}{3} = 2$.

There are a total $\frac{6!}{2!2!2!} - 6 = 90 - 6 = 84$ permutations of the form 6-6-6. Correcting for repetition, we get $\frac{84}{6} = 14$.

There are a total $\frac{9!}{3!3!3!} - 6 = 1680 - 6 = 1674$ permutations of the form 9-9. Correcting for repetition, we get $\frac{1674}{9} = 186$.

The remaining permutations are of the form 18, and there are $s = \frac{18!}{6!6!6!} - (6 + 84 + 1674)$ of them. Simplifying, we get $s = \frac{18 \cdot 17 \cdot \ldots \cdot 7}{6!6!} - (6 + 84 + 1674) = 18 \cdot 17 \cdot 4 \cdot 14 \cdot 13 \cdot 11 \cdot 7 - 1764$. Correcting for repetition, we get $\frac{s}{18} = 17 \cdot 4 \cdot 14 \cdot 13 \cdot 11 \cdot 7 - 98$.

Recall that $13 \cdot 11 \cdot 7 = 1001 \equiv 1 \pmod{1000}$. So, our answer is

$2 + 14 + 186 + 17 \cdot 4 \cdot 14 - 98 = 1056 \equiv \boxed{56} \pmod{1000}$.

13. [**9**] There is a right triangle $\triangle ABC$, in which $\angle A$ is the right angle. On side AB, there are three points X, Y, and Z that satisfy $\angle ACX = \angle XCY = \angle YCZ = \angle ZCB$ and $BZ = 2AX$. The smallest angle of $\triangle ABC$ is $\frac{a}{b}$ degrees, where a, b are positive integers such that $GCD(a, b) = 1$. Find $a + b$.

Solution:

Without loss of generality, let $AX = 1$. Let $XY = x$, $YZ = y$, $CA = h, CX = p, CY = q, CZ = r, CB = s$. Reflect the triangle ABC (and all additional points defined on its perimeter) across line AC to get a triangle BCD, with the perpendicular foot from C to BD being A.

By angle bisector theorem, $x : y = p : r = 2 : x + y$

and

$y : 2 = q : s = x + 2 : x + y + 2$

From these, one can get two equations: 2y = x(x + y) and 2(x + 2) = y(x + y + 2). Solving, we get $x^3 - 4x^2 - 4^x + 8 = 0$.

Again, by angle bisector theorem, $1 : x = h : q$. Apply Pythagorean Theorem to get $(hx)^2 = q^2 = h^2 + (1 + x)^2$. So $h^2 = \frac{1+x}{1-x}$.

I claim that $p = x + y + 2$. Using Pythagorean theorem, one gets $p^2 = h^2 + 1$. Applying this, it is easy to show that proving $p^2 = (x+y+2)^2$ reduces to proving $x(2-x)^2 = 8(1-x)$, which when simplified becomes $x^3 - 4x^2 - 4^x + 8 = 0$, which we have already shown to be true. So, $p = x+y+2$. So, $CX = XB$. So, XBC is a isosceles triangle, so $\angle B = \angle XCB = \frac{3}{4}\angle C$. Then the smallest angle is $\angle B$, and from the above relationship, one gets that $\angle B = \frac{3}{7} \cdot 90 = \frac{270}{7}$. So the answer is $\boxed{277}$.

14. [**9**] Define function $f_k(x)$ (where k is a positive integer) as follows:

$$f_k(x) = (\cos kx)(\cos x)^k + (\sin kx)(\sin x)^k - (\cos 2x)^k$$

Find the sum of all distinct value(s) of k such that $f_k(x)$ is a constant function.

Solution:

Since $f_k(0) = 0$, if f_k is a constant function, then it must be identically equal to 0. Furthermore, it must be true that $f_k(\frac{\pi}{2}) = \sin\frac{k\pi}{2} - (-1)^k = 0$. It follows that $k \equiv 3 \pmod 4$. Let $k = 4n - 1$. Then, $f_k(x) = -\cos^{4n-1}(\frac{\pi}{4n-1}) - \cos^{4n-1}(\frac{2\pi}{4n-1}) = 0$. From this, one can obtain $\cos\frac{\pi}{4n-1} + \cos\frac{2\pi}{4n-1} = 0$, $(2\cos\frac{\pi}{4n-1} - 1)(\cos\frac{2\pi}{4n-1} + 1) = 0$, and $\cos\frac{\pi}{4n-1} = \frac{1}{2}$. So, it is necessary that $n = 1$, which means it is necessary that $k = 3$. And to show that this is sufficient, check that $f_3(x) = 0$. So the answer is $\boxed{3}$.

15. [**10**] Jason has n coins, among which at most one of them is counterfeit. The counterfeit coin (if there is any) is either heavier or lighter than a real coin. Jason's grandfather also left him an old weighing balance, on which he can place any number of coins on either side and the balance will show which side is heavier. However, the old weighing balance is in fact really really old and can only be used 4 more times. What is the largest number n for which is it possible for Jason to find the counterfeit coin (if it exist)?

Solution:

For $n = 39$, we split it into 3 groups of 13. Let the coins from the groups be group A, B, C coins. We weight $13A$ against $13B$.

Case 1: $13A = 13B$

Hence the fake can only be among $13C$ and the other 26 coins are reals. Hence we weigh $9C$ against $9A$.

Case 1a: $9C = 9A$. Hence the fake can only be among the remaining $4C$. We weigh $3C$ agaisnt $3A$ If equal, then only the last C coin can be fake, which can be checked by weighing against a A coin. If $3C > 3A$, then take any two of the $3C$ and weigh them. The heavier is fake and if they are the same, the last C coin of the three is fake. $3C < 3A$ is similar.

Case 1b: $9C < 9A$ Hence the fake is among the $9C$. Take two groups of $3C$ and weigh them. The fake is among the lighter group and if they are equal, the fake is among the remaining $3C$. Hence we take any two of the set of $3C$ and weigh them. The lighter is fake and if they are the same, the last C coin of the three is fake.

Case 1c: $9C > 9A$ This is similar to case 1b.

Case 2: $13A < 13B$.

Hence the $13C$ coins are all real. Weigh $9A + 4B$ agaisnt $4A + 9C$.

Case 2a: $9A + 4B = 4A + 9C$ Hence the fake is among the remaining $9B$ and heavier, making this similar to case 1c.

Case 2b: $9A + 4B < 4A + 9C$ Hence the fake is among the remaining $9A$ and lighter, making this similar to case 1b.

Case 2c: $9A + 4B > 4A + 9C$ Hence the fake is among the $4B$ on the LHS or the $4A$ on the RHS. We weight $3A + 3B$ against $6C$. If equal, then the fake is among the remaining A, B which can be easily identified by weighing A with C. If $3A + 3B > 6C$, the fake is heavier and among the $3B$. Otherwise, it is lighter and among $3A$. Either of which case can be checked just as in case 1a.

Case 3: $13A > 13B$

Similar to case 2.

We see that for $n = 40$, since there are 81 scenarios, with each of the 40 coins being heavier or lighter, or all being real, and there are 4 weighings, with 3 outcomes each, for a total of 81 outcomes, we need each weighing to eliminate the possible outcomes down to 1/3. Otherwise, if scenarios is larger than outcomes, it will be impossible for the fake coin to be found with certainty.

Let the first weighing be between x coins and x coins. If $x \leq 13$, in the case that the two piles of x coins are equal, the fake can only be among the remaining $40 - 2x$ coins, which give rise to $2(40 - 2x) + 1 = 81 - 4x \geq 29$ scenarios. If $x \geq 14$, then in the case that one of the x coins is heavier than the other, there will be $2x \geq 28$ scenarios.

Thus in all cases, there might be more scenarios than the 27 outcomes possible with 3 weighing remaining. Hence it is impossible to always find the fake coin for $n = 40$, and $n = \boxed{39}$ is the maximum possible.