PUMaC 2015

Algebra A

1. [**3**] How many integer pairs (a, b) with $1 < a, b \leq 2015$ are there such that $\log_a b$ is an integer?

2. [**3**] There are real numbers a, b, c, d such that for all (x, y) satisfying $6y^2 = 2x^3 + 3x^2 + x$, if $x_1 = ax + b$ and $y_1 = cy + d$, then $y_1^2 = x_1^3 - 36x_1$. What is $a + b + c + d$?

3. [**4**] Find the sum of the non-repeated roots of the polynomial $P(x) = x^6 - 5x^5 - 4x^4 - 5x^3 + 8x^2 + 7x + 7$.

4. [**4**] Define the sequence a_i as follows: $a_1 = 1, a_2 = 2015$, and $a_n = \frac{na_{n-1}^2}{a_{n-1}+na_{n-2}}$ for $n > 2$. What is the least k such that $a_k < a_{k-1}$?

5. [**5**] Since counting the numbers from 1 to 100 wasn't enough to stymie Gauss, his teacher devised another clever problem that he was sure would stump Gauss. Defining $\zeta_{15} = e^{2\pi i/15}$ where $i = \sqrt{-1}$, the teacher wrote the 15 complex numbers ζ_{15}^k for integer $0 \leq k < 15$ on the board. Then, he told Gauss:

   On every turn, erase two random numbers a, b, chosen uniformly randomly, from the board and then write the term $2ab - a - b + 1$ on the board instead. Repeat this until you have one number left. What is the expected value of the last number remaining on the board?

6. [**6**] We define the function $f(x, y) = x^3 + (y - 4)x^2 + (y^2 - 4y + 4)x + (y^3 - 4y^2 + 4y)$. Then choose any distinct $a, b, c \in \mathbb{R}$ such that the following holds: $f(a, b) = f(b, c) = f(c, a)$. Over all such choices of a, b, c, what is the maximum value achieved by:

   $$\min(a^4 - 4a^3 + 4a^2, b^4 - 4b^3 + 4b^2, c^4 - 4c^3 + 4c^2)?$$

7. [**7**] We define the *ridiculous* numbers recursively as follows:

   (a) 1 is a *ridiculous* number.

   (b) If a is a *ridiculous* number, then $\sqrt{a}$ and $1 + \sqrt{a}$ are also *ridiculous* numbers.

   A closed interval I is *boring* if

   - I contains no *ridiculous* numbers, and
   - There exists an interval $[b, c]$ containing I for which b and c are both *ridiculous* numbers.

   The smallest non-negative l such that there does not exist a *boring* interval with length l can be represented in the form $\dfrac{a + b\sqrt{c}}{d}$ where a, b, c, d are integers, $\gcd(a, b, d) = 1$, and no integer square greater than 1 divides c. What is $a + b + c + d$?

8. [**8**] Let $P(x)$ be a polynomial with positive integer coefficients and degree 2015. Given that there exists some $\omega \in \mathbb{C}$ satisfying:

   $$\omega^{73} = 1 \quad \text{and}$$

   $$P(\omega^{2015}) + P(\omega^{2015^2}) + P(\omega^{2015^3}) + \ldots + P(\omega^{2015^{72}}) = 0,$$

   what is the minimum possible value of $P(1)$?