Algebra A

1. [**3**] How many integer pairs (a, b) with $1 < a, b \leq 2015$ are there such that $\log_a b$ is an integer?

 Solution:

 We do casework on how large $\log_a b$ is. Note that $\log_a b > 0$ since $b > 1$ and $a \geq 2$ while $b \leq 2015$ so $\log_a b \leq \log_2 2015 < 11$.

 If $\log_a b = 1$, then $a = b$ so there are 2014 such pairs.

 If $\log_a b = 2$, then $a^2 = b \leq 2015$ so $a < 45$ so there are 43 such pairs.

 If $\log_a b = 3$, then $a^3 = b \leq 2015$ so $a < 13$ so there are 11 such pairs. Note that we can find this bound easily by $12^3 = 144 \cdot 12 < 150 \cdot 12 = 1800$ and $13^3 = 169 \cdot 13 > 160 \cdot 13 = 2080$.

 We continue for each integer value of $\log_a b$ less than 11 and in total there are $2014 + 43 + 11 + 5 + 3 + 2 + 1 + 1 + 1 + 1 = \boxed{2082}$ such pairs.

 Author: Roy Zhao

2. [**3**] There are real numbers a, b, c, d such that for all (x, y) satisfying $6y^2 = 2x^3 + 3x^2 + x$, if $x_1 = ax + b$ and $y_1 = cy + d$, then $y_1^2 = x_1^3 - 36x_1$. What is $a + b + c + d$?

 Solution:

 We have $6y^2 = 2x^3 + 3x^2 + x$. Make the following substitution: $y = 3y'$, $x = 3x'$. This gives $y'^2 = x'^3 + \frac{1}{2}x'^2 + \frac{1}{18}x'$. Further, substitute $x'' = x' + \frac{1}{6}$. This gives

$$y'^2 = x''^3 - \frac{1}{36}x''$$
$$36^3 y'^2 = 36^3 x''^3 - 36^2 x''$$
$$(6^3 y')^2 = (36x'')^3 - 36(36x'').$$

 Finally, make the substitution $y_1 = 216y'$ and $x_1 = 36x''$. Thus $y_1^2 = x_1^3 - 36x_1$, and by substitution, $y_1 = 72y$ and $x_1 = 12x + 6$. So our answer is $12 + 6 + 72 + 0 = \boxed{90}$.

 EDIT: This problem did not specify that a and c should be non-zero. As such, any solution using $a = c = 0$ and (b, d) a solution to $y_1^2 = x_1^3 - 36x_1$ was valid. This problem has been thrown out due to such a wide variety of possible answers. *Author: Heesu Hwang*

3. [**4**] Find the sum of the non-repeated roots of the polynomial $P(x) = x^6 - 5x^5 - 4x^4 - 5x^3 + 8x^2 + 7x + 7$.

 Solution:

 Note that $P(x) = (x^6 - 5x^5 - 4x^4 - 5x^3 + x^2) + (7x^2 + 7x + 7) = x^2(x^4 - 5x^3 - 4x^2 - 5x + 1) + 7(x^2 + x + 1)$. We check that $(x^4 - 5x^3 - 4x^2 - 5x + 1) = (x^2 + x + 1)(x^2 - 6x + 1)$. Thus $P(x) = (x^2 + x + 1)(x^2(x^2 - 6x + 1) + 7) = (x^2 + x + 1)(x^4 - 6x^3 + x^2 + 7)$. There are still no convenient roots of the quartic, thus we again use standard guess-and-check with quadratics of constant terms 7 and 1 to find $(x^2 + x + 1)(x^2 - 7x + 7) = x^4 - 6x^2 + x^2 + 7 \implies P(x) = (x^2 + x + 1)^2(x^2 - 7x + 7)$. The distinct roots are those of $(x^2 - 7x + 7)$, and thus by Vieta's, the sum of the distinct roots is $\boxed{7}$.

 Author: Christopher Zhang

4. [**4**] Define the sequence a_i as follows: $a_1 = 1, a_2 = 2015$, and $a_n = \frac{na_{n-1}^2}{a_{n-1}+na_{n-2}}$ for $n > 2$. What is the least k such that $a_k < a_{k-1}$?

   Solution:

   The recursion is equivalent to $\frac{a_{n-1}}{a_n} = \frac{a_{n-2}}{a_{n-1}} + \frac{1}{n} = \frac{1}{2015} + \sum_{i=3}^{n} \frac{1}{i}$. The first k for which $\frac{a_{k-1}}{a_k} > 1$ occurs when $k = \boxed{7}$ by a simple computation of the sum.

   Author: Bill Huang

5. [**5**] Since counting the numbers from 1 to 100 wasn't enough to stymie Gauss, his teacher devised another clever problem that he was sure would stump Gauss. Defining $\zeta_{15} = e^{2\pi i/15}$ where $i = \sqrt{-1}$, the teacher wrote the 15 complex numbers ζ_{15}^k for integer $0 \le k < 15$ on the board. Then, he told Gauss:

   On every turn, erase two random numbers a, b, chosen uniformly randomly, from the board and then write the term $2ab - a - b + 1$ on the board instead. Repeat this until you have one number left. What is the expected value of the last number remaining on the board?

   Solution:

   If we let the operation $\diamond$ be defined as $a \diamond b = 2ab - a - b + 1$, then it can be shown that $\diamond$ is associative and commutative. Therefore we will get the same value regardless of how we erase the elements from the board. Then given an arbitrary sequence of numbers $a_1, a_2, \cdots, a_n$, it is not hard to show by induction that:

   $$a_1 \diamond a_2 \diamond \cdots \diamond a_n = 2^{n-1} \prod_{i=1}^{n} (a_i - 1/2) + 1/2$$

   Therefore we have that:

   $$\zeta_{15}^0 \diamond \zeta_{15}^1 \diamond \cdots \diamond \zeta_{15}^{14} = 2^{14} \prod_{k=0}^{14} (\zeta_{15}^k - 1/2) + 1/2$$

   But we can factor:

   $$1 - x^{15} = \prod_{k=0}^{14} (\zeta_{15}^k - x)$$

   And hence our answer is:

   $$2^{14} \prod_{k=0}^{14} (\zeta_{15}^k - 1/2) + 1/2 = 2^{14}(1 - (1/2)^{15}) + 1/2 = 2^{14} = \boxed{16384}$$

   Author: Roy Zhao

6. [**6**] We define the function $f(x, y) = x^3 + (y-4)x^2 + (y^2 - 4y + 4)x + (y^3 - 4y^2 + 4y)$. Then choose any distinct $a, b, c \in \mathbb{R}$ such that the following holds: $f(a, b) = f(b, c) = f(c, a)$. Over all such choices of a, b, c, what is the maximum value achieved by:

   $$\min(a^4 - 4a^3 + 4a^2, b^4 - 4b^3 + 4b^2, c^4 - 4c^3 + 4c^2)?$$

Solution:

Let $f(x) = x^4 - 4x^3 + 4x^2$ then the equalities become $\frac{f(b) - f(a)}{b - a} = \frac{f(c) - f(b)}{c - b} = \frac{f(a) - f(c)}{a - c}$. So this implies that the points $(a, f(a)), (b, f(b)), (c, f(c))$ lie on the same line and the question now becomes: what is the maximum, over all such lines which intersect $f(x)$ at least 3 times, of the 3rd highest intersection point?

We claim that the highest intersection points occurs when the line is horizontal and tangent to $f(x)$ at $x = 1, f(x) = 1$. This is simply because this is a valid point and if we choose a, b such that $f(a), f(b) > 1$, then it is clear from looking at the graph of $f(x)$ that the line between $(a, f(a)), (b, f(b))$ can only intersect $f(x)$ two times. Therefore $f(x) = \boxed{1}$ is our answer.

Author: Roy Zhao

7. [**7**] We define the *ridiculous* numbers recursively as follows:

   (a) 1 is a *ridiculous* number.

   (b) If a is a *ridiculous* number, then $\sqrt{a}$ and $1 + \sqrt{a}$ are also *ridiculous* numbers.

   A closed interval I is *boring* if

   - I contains no *ridiculous* numbers, and
   - There exists an interval $[b, c]$ containing I for which b and c are both *ridiculous* numbers.

   The smallest non-negative l such that there does not exist a *boring* interval with length l can be represented in the form $\frac{a + b\sqrt{c}}{d}$ where a, b, c, d are integers, $\gcd(a, b, d) = 1$, and no integer square greater than 1 divides c. What is $a + b + c + d$?

   Solution:

   The smallest ridiculous number is 1. This is true because if $a \geq 1$, then $\sqrt{a} \geq 1$ and $1 + \sqrt{a} \geq 1$. The supremum of the ridiculous numbers (the smallest number that is greater than all ridiculous numbers) is $\frac{3+\sqrt{5}}{2}$. This is because the largest ridiculous number that is n recursive steps away from 1 is $1 + \sqrt{1 + \sqrt{1 + \ldots\sqrt{1}}}$, where there are n square root signs. As n approaches infinity, the largest ridiculous numbers approach $M = 1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$. It can be seen that $M = \frac{3+\sqrt{5}}{2}$ by observing that m satisfies the identity $(M - 1)^2 = M$.

   There are no ridiculous numbers in the interval $[\frac{1+\sqrt{5}}{2}, 2)$: In the case that r is of the form $1 + \sqrt{a}$ for some ridiculous a, then r must be at least 2, because a must be at least 1. In the case that r is of the form $\sqrt{a}$, then r must be less than $\sqrt{\frac{3+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2}$. Since these are the only two cases, if $I = [\frac{1+\sqrt{5}}{2}, 2 - \epsilon]$ for some small $\epsilon > 0$, then I is a boring interval, so l must be at least $2 - \frac{1+\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2}$.

   Now we show that there does not exist a boring interval of length $\frac{3-\sqrt{5}}{2}$. Assume for the sake of contradiction that such an interval exists. It must be contained in $[1, \frac{1+\sqrt{5}}{2}]$ or $[2, \frac{3+\sqrt{5}}{2}]$. The first case is ruled out because $\sqrt{\sqrt{2}}$ and $\sqrt{2}$ are ridiculous numbers. The second case is ruled out because $1+\sqrt{\sqrt{2}}$ and $1+\sqrt{2}$ are ridiculous. Therefore $l=\frac{3-\sqrt{5}}{2}$ and our answer is $3+(-1)+5+2=\boxed{9}$.

Author: Ben Edelman

8. [**8**] Let $P(x)$ be a polynomial with positive integer coefficients and degree 2015. Given that there exists some $\omega \in \mathbb{C}$ satisfying:

$$\omega^{73}=1 \quad \text{and}$$
$$P(\omega^{2015})+P(\omega^{2015^2})+P(\omega^{2015^3})+\ldots+P(\omega^{2015^{72}})=0,$$

what is the minimum possible value of $P(1)$?

Solution:

Note that $\omega \neq 1$ since $P(1)>0$ by the condition that the coefficients are all positive. Let $P(x)=a_{2015}x^{2015}+\cdots+a_0x^0$. Now, we claim that 2015 is a primitive root mod 73. It suffices to show that $2015^{24}, 2015^{36}$ are not equal to 1 mod 73. The latter follows by quadratic reciprocity and the first can be calculated to be equal to -9 mod 73.

Now, the given expression is equivalent to:

$$\sum_{k=1}^{72} P(\omega^k)=\sum_{k=1}^{73} P(\omega^k)-P(1)=73\cdot(a_0x^0+a_{73}x^{73}+\ldots+a_{27\cdot 73}x^{27\cdot 73})-P(1)=0$$

And since $a_i \geq 1$, we have:

$$P(1)=73\cdot(a_0x^0+a_{73}x^{73}+\ldots+a_{27\cdot 73}x^{27\cdot 73})\geq 73\cdot 28=2044$$

Thus, the minimum possible value of $P(1)$ is $\boxed{2044}$.

Author: Bill Huang