Combinatorics A

1. [**3**] A word is an ordered, non-empty sequence of letters, such as *word* or *wrod*. How many distinct words can be made from a subset of the letters c, o, m, b, o, where each letter in the list is used no more than the number of times it appears?

2. [**3**] Andrew has 10 balls in a bag, each a different color. He randomly picks a ball from the bag 4 times, with replacement. The expected number of distinct colors among the balls he picks is $\frac{p}{q}$, where $\gcd(p, q) = 1$ and $p, q > 0$. What is $p + q$?

3. [**4**] Consider a random permutation of the set $\{1, 2, \ldots, 2015\}$. In other words, for each $1 \leq i \leq 2015$, i is sent to the element a_i where $a_i \in \{1, 2, \ldots, 2015\}$ and if $i \neq j$, then $a_i \neq a_j$. What is the expected number of ordered pairs (a_i, a_j) with $i - j > 155$ and $a_i - a_j > 266$?

4. [**4**] A number is *interesting* if it is a 6-digit integer that contains no zeros, its first 3 digits are strictly increasing, and its last 3 digits are non-increasing. What is the average of all interesting numbers?

5. [**5**] Alice has an orange 3-by-3-by-3 cube, which is comprised of 27 distinguishable, 1-by-1-by-1 cubes. Each small cube was initially orange, but Alice painted 10 of the small cubes completely black. In how many ways could she have chosen 10 of these smaller cubes to paint black such that every one of the 27 3-by-1-by-1 sub-blocks of the 3-by-3-by-3 cube contains at least one small black cube?

6. [**6**] Every day, Heesu talks to Sally with some probability p. One day, after not talking to Sally the previous day, Heesu resolves to ask Sally out on a date. From now on, each day, if Heesu has talked to Sally each of the past four days, then Heesu will ask Sally out on a date. Heesu's friend remarked that at this rate, it would take Heesu an expected 2800 days to finally ask Sally out. Suppose $p = \frac{m}{n}$, where $\gcd(m, n) = 1$ and $m, n > 0$. What is $m + n$?

7. [**7**] The lattice points (i, j) for integers $0 \leq i, j \leq 3$ are each being painted orange or black. Suppose a coloring is *good* if for every set of integers x_1, x_2, y_1, y_2 such that $0 \leq x_1 < x_2 \leq 3$ and $0 \leq y_1 < y_2 \leq 3$, the points $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)$ are not all the same color. How many good colorings are possible?

8. [**8**] In a tournament with 2015 teams, each team plays every other team exactly once and no ties occur. Such a tournament is *imbalanced* if for every group of 6 teams, there exists either a team that wins against the other 5 or a team that loses to the other 5. If the teams are indistinguishable, what is the number of distinct imbalanced tournaments that can occur?