Combinatorics A Solutions

1. [**3**] A word is an ordered, non-empty sequence of letters, such as *word* or *wrod*. How many distinct words can be made from a subset of the letters c, o, m, b, o, where each letter in the list is used no more than the number of times it appears?

   Solution: We use casework on the length of the word.

   - There are 4 one-letter words: $\{c, o, m, b\}$.
   - There is 1 two-letter word with 2 o's and $4 \cdot 3 = 12$ two-letter words with at most one o for a total of 13.
   - There are $\binom{3}{2} \cdot 3 = 9$ three-letter words with 2 o's (we choose the positions of the o's and then choose the third letter) and $4 \cdot 3 \cdot 2 = 24$ for a total of 33.
   - Note that the number of four-letter words is equal to the number of five-letter words, as the last letter is determined by the first four, for a total of $\frac{5!}{2!} = 60$ each.

   Summing, we obtain a total of $\boxed{170}$ words.

   Author: Bill Huang

2. [**3**] Andrew has 10 balls in a bag, each a different color. He randomly picks a ball from the bag 4 times, with replacement. The expected number of distinct colors among the balls he picks is $\frac{p}{q}$, where $\gcd(p, q) = 1$ and $p, q > 0$. What is $p + q$?

   Solution: The probability that any particular one of the 10 colors is picked is $p = 1 - \left(\frac{9}{10}\right)^4 = \frac{3439}{10000}$. The expected contribution towards the total number of distinct colors picked by any particular color is then $p \cdot 1 + (1 - p) \cdot 0 = p$, and by linearity of expectation, since we have 10 colors, the expected total number of distinct colors is $E = 10 \cdot p = \frac{3439}{1000}$, so $p = 3439$ and $q = 1000$ and $p + q = \boxed{4439}$.

   Author: Roy Zhao

3. [**4**] Consider a random permutation of the set $\{1, 2, \ldots, 2015\}$. In other words, for each $1 \le i \le 2015$, i is sent to the element a_i where $a_i \in \{1, 2, \ldots, 2015\}$ and if $i \neq j$, then $a_i \neq a_j$. What is the expected number of ordered pairs (a_i, a_j) with $i - j > 155$ and $a_i - a_j > 266$?

   Solution: First, observe that the total number of ordered pairs a_i, a_j satisfying $i - j > 155$ is equal to $(2015 - 156) + (2015 - 157) + \ldots + 1 = \binom{2015-155}{2} = 1728870$, where we count by casework on $j = 1, 2, \ldots, 1859$.

   Since the permutation is random, the probability that any arbitrary ordered pair of elements a_i, a_j satisfy $a_i - a_j > n$ for some n is the same for any i, j. Furthermore, this probability is equal to the number of ordered pairs x, y satisfying $x - y > n$ and the total number of ordered pairs (x, y), as (a_i, a_j) is equally likely to be any ordered pair x, y (the former counted similarly as above). Thus, when $n = 266$ the probability is:

$$p = \frac{\binom{2015-266}{2}}{2015 \cdot 2014} = \frac{759}{2015}$$

By linearity of expectation, we have that the expected total number of ordered pairs a_i, a_j satisfying $i - j > 155$ that also satisfy $a_i - a_j > 266$ is:

$$E = 1728870 \cdot \frac{759}{2015} = \boxed{651222}$$

Authors: Roy Zhao, Bill Huang

4. [**4**] A number is *interesting* if it is a 6-digit integer that contains no zeros, its first 3 digits are strictly increasing, and its last 3 digits are non-increasing. What is the average of all interesting numbers?

Solution: We calculate the expected value of each digit, then use linearity of expectation to find the total expected value. Let $a_1, a_2, a_3, a_4, a_5, a_6$ denote the expected value of each digit and let the expected value be $a = \overline{a_1a_2a_3a_4a_5a_6}$. By symmetry, we have $a_2 = a_5 = 5$. and $a_1 + a_3 = a_4 + a_6 = 10$.

If $a_1 = k$, then $a_2, a_3 \in \{k+1, \ldots, 9\}$ and (a_2, a_3) correspond to the number of ways:

$$(9 - k - 1) + (9 - k - 2) + \ldots + 1 = \binom{9-k}{2}$$

To choose two numbers among $\{k+1, \ldots, 9\}$, so calculating and using the hockey stick identity:

$$a_1 = \frac{1 \cdot \binom{8}{2} + 2 \cdot \binom{7}{2} + \ldots + 7 \cdot \binom{2}{2}}{\binom{8}{2} + \binom{7}{2} + \ldots + \binom{2}{2}} = \frac{\binom{9}{3} + \binom{8}{3} + \ldots + \binom{3}{3}}{\binom{9}{3}} = \frac{\binom{10}{4}}{\binom{9}{3}} = \frac{10}{4} = \frac{5}{2}$$

If $a_6 = k$, then $a_5, a_4 \in \{k, \ldots, 9\}$ and (a_5, a_4) correspond to the number of ways:

$$(9 - k + 1) + (9 - k) + \ldots + 1 = \binom{9-k+2}{2}$$

To choose two numbers among $\{k, \ldots, 9\}$ allowing replacement, so calculating and using the hockey stick identity:

$$a_6 = \frac{1 \cdot \binom{10}{2} + 2 \cdot \binom{9}{2} + \ldots + 9 \cdot \binom{2}{2}}{\binom{10}{2} + \binom{9}{2} + \ldots + \binom{2}{2}} = \frac{\binom{11}{3} + \binom{10}{3} + \ldots + \binom{3}{3}}{\binom{11}{3}} = \frac{\binom{12}{4}}{\binom{11}{3}} = \frac{12}{4} = 3$$

Then it follows that $a = \overline{a_1a_2a_3a_4a_5a_6} = 250000 + 50000 + 7500 + 700 + 50 + 3 = \boxed{308253}$.

Authors: Victor Zhou, Bill Huang

5. [**5**] Alice has an orange 3-by-3-by-3 cube, which is comprised of 27 distinguishable, 1-by-1-by-1 cubes. Each small cube was initially orange, but Alice painted 10 of the small cubes completely black. In how many ways could she have chosen 10 of these smaller cubes to paint black such that every one of the 27 3-by-1-by-1 sub-blocks of the 3-by-3-by-3 cube contains at least one small black cube?

Solution: Divide the 3-by-3-by-3 cube into 3 1-by-3-by-3 blocks. If 10 total smaller cubes are painted black, then two of these blocks must contain 3 black cubes and the third contains

4. Now, if a black does not have a diagonal of black cubes (allowing wrap-arounds), it must contain at least 4 cubes, so there are at least two blocks with diagonals and with exactly 3 cubes. We consider two cases.

<u>Case 1:</u> The diagonals of these two blocks are oriented in the same direction.

Clearly, the third block must contain a diagonal oriented in the same direction as well. The remaining black cube can be anywhere else in the block. There are $3 \cdot 6 \cdot 2 = 36$ ways to choose the first two blocks and their diagonals. There are $1 \cdot 6 = 6$ ways to choose black cubes in the remaining block. This gives a total of 216 colorings.

<u>Case 2:</u> They are oriented in opposite directions.

Then, the black cubes in the remaining block is determined (consider the projection of the blocks on top of one another; four squares are missing and the remaining block contains four black cubes). There are $3 \cdot 6 \cdot 3 = 54$ ways to choose the first to blocks and their diagonals. There is only 1 way to choose the black cubes in the remaining block. This gives a total of 54 colorings.

In total, then, there are $216 + 54 = \boxed{270}$ ways to choose 10 of the smaller cubes to paint black.

Author: Bill Huang

6. [**6**] Every day, Heesu talks to Sally with some probability p. One day, after not talking to Sally the previous day, Heesu resolves to ask Sally out on a date. From now on, each day, if Heesu has talked to Sally each of the past four days, then Heesu will ask Sally out on a date. Heesu's friend remarked that at this rate, it would take Heesu an expected 2800 days to finally ask Sally out. Suppose $p = \frac{m}{n}$, where $\gcd(m, n) = 1$ and $m, n > 0$. What is $m + n$?

Solution: Let E be the expected number of days it would take Heesu to ask Sally out (in terms of p). Let E_k for $k \in \mathbb{Z}$ denote the expected number of days it would take given that Heesu has talked to Sally each of the last k days but not the $(k+1)$-th day before. Then, for all $k \geq 4$, $E_k = 0$. Additionally, we have:

$$\begin{aligned} E_0 &= 1 + (1-p)E_0 + pE_1 \\ E_1 &= 1 + (1-p)E_0 + pE_2 \\ E_2 &= 1 + (1-p)E_0 + pE_3 \\ E_3 &= 1 + (1-p)E_0 + pE_4 \end{aligned}$$

Subtracting the equations, we then have:

$$\frac{1}{p} = E_0 - E_1 = p(E_1 - E_2) = p^2(E_2 - E_3) = p^3(E_3 - E_4) = p^3 E_3$$

Then, solving, we obtain:

$$E_3 = \frac{1}{p^4}, \quad E_2 = \frac{1}{p^3} + \frac{1}{p^4}, \quad E_1 = \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4}, \quad E_0 = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4}$$

Since $E = E_0$, we have:

$$2800 = 7 + 49 + 343 + 2401 = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4}$$

So $p = \frac{1}{7}$ and $m+n = \boxed{8}$.

Author: Roy Zhao

7. [**7**] The lattice points (i,j) for integer $0 \le i,j \le 3$ are each being painted orange or black. Suppose a coloring is *good* if for every set of integers x_1, x_2, y_1, y_2 such that $0 \le x_1 < x_2 \le 3$ and $0 \le y_1 < y_2 \le 3$, the points $(x_1,y_1),(x_1,y_2),(x_2,y_1),(x_2,y_2)$ are not all the same color. How many good colorings are possible?

A 4×4 square grid of lattice points (i,j) for $0 \le i,j \le 3$ have its lattice points painted orange or black. How many colors are possible such that for every

Solution: First, it is not possible for there to be two rows or columns with 3 black tiles or two rows or columns with 3 orange tiles. And by this logic, we cannot have a row or column all of the same color. So the maximum number of orange tiles is $3+2+2+2=9$ and same for black. So there are $7, 8$ or 9 orange tiles. But for every valid board with 7 orange tiles, if we change all the tiles colors, we will have a valid board with 9 orange tiles. Therefore we will count how many valid boards there are with 8 orange tiles and 9 orange tiles, and the number of valid board with 7 orange tiles is the same as 9 orange tiles.

Now we note that if we rearrange the rows of a valid grid, we will still end up with a valid grid. And we also know that for a valid grid, we cannot have two identical rows and hence for each valid grid, we can rearrange the 4 rows $4! = 24$ different ways to get other valid grids. It is also possible to rearrange columns but we will not count those to prevent double counting.

Case 1: 8 orange tiles. There are two possible arrangements of the 8 orange tiles for the rows, $3-2-2-1$ and $2-2-2-2$. For the first arrangement, there are 4 ways to select 3 orange tiles in a row. Then for the next two rows with both 2 tiles, they must have an orange tile where the first row had a black tile and hence there are 3 ways to choose these two rows. Then for the last row, there is only one way to place the orange tile to prevent there being a rectangle formed by 4 black tile and there will be no rectangle with black tiles or orange tiles if we choose tiles this way. So there are a total of $4 \cdot 3 = 12$ ways to select the rows and 24 ways to rearrange them. For example:

$$\begin{matrix} o & o & o & b \\ o & b & b & o \\ b & o & b & o \\ b & b & o & b \end{matrix}$$

Now for the $2-2-2-2$ case, it is easy to verify that no matter how we choose two orange tiles per row, as long as the rows are distinct, we will not get any rectangles. Therefore there are $\binom{\binom{4}{2}}{4} = 15$ different ways to select the rows and 24 ways to rearrange them.

Case 2: 9 orange tiles. There is only one possible way to have 9 orange tiles, and that is $3-2-2-2$. There are 4 ways to choose 3 orange tiles for the first row and then the 3 remaining rows with 2 tiles must have an orange tile where the first row had a black tile and so there is only 1 way to select these three rows up to permutation. So there are 4 ways to select the rows and again 24 ways to permute them. Therefore our answer is $15 \cdot 24 + 12 \cdot 24 + 4 \cdot 24 + 4 \cdot 24 = 35 \cdot 24 = \boxed{840}$.

Author: Roy Zhao

8. [**8**] In a tournament with 2015 teams, each team plays every other team exactly once and no ties occur. Such a tournament is *imbalanced* if for every group of 6 teams, there exists either a team that wins against the other 5 or a team that loses to the other 5. If the teams are indistinguishable, what is the number of distinct imbalanced tournaments that can occur?

Solution: We claim that within any subset S of T with $|S| \geq 6$, there must exist a winner or a loser (i.e. someone who wins against or loses to everyone else in that subset). We show this by induction, and by assumption, it is already true for the base case $|S| = 6$, so assume $|S| > 6$. Assume the contrary, that in S, every team wins and loses at least 1 game. Let the set of teams that win exactly 1 game be A and that lose exactly 1 game be B.

Now, each team v must either win against at least one team that loses exactly 1 game or lose to at least one team that wins exactly 1 game. Otherwise, take $S - v$, and by the inductive hypothesis, there must exist a winner or a loser. Since the winner or loser of $S - v$ does not lose or win, respectively, to v, that team must still be a winner or loser, respectively, of S. Then, counting $|A|$ and $|B|$, we have:

$$|A| + |B| \geq |S| > 6$$

This means that either $|A| \geq 4$ or $|B| \geq 4$. If $A \geq 4$, then there must be some team $a \in A$ that wins against at least 2 teams in A, which contradicts the definition of teams in A, so $A \geq 4$ is impossible. Similarly, $B \geq 4$ is impossible. Thus, a contradiction is reached, so there must exist a team in S that is either a winner or a loser, which completes the inductive step.

We now count the number of distinct imbalanced tournaments. Given any imbalanced tournament T, we proceed to repeatedly select either a winner or a loser until we run out of options. We end up with an ordering of all 2015 teams with the exception of a leftover group G of at most 5 teams. For instance, we could have:

$$a_1 > a_2 > \ldots > a_{420} > (a_{421}, a_{422}, a_{423}, a_{424}, a_{425}) > a_{426} > \ldots > a_{2015}$$

If $|G| = 0$, then the only possible tournament is $a_1 > a_2 > \ldots > a_{2015}$, so there is 1 distinct tournament. Note that $|G| \neq 1, 2$, as we can always select one team from that group that wins against every other team from that group.

For the sake of convenience, let L be the set of teams that lose to G. If $|G| = 3$, the only distinct $G = \{a_i, a_j, a_k\}$ is WLOG $a_i \to a_j; a_j \to a_k; a_k \to a_i$. Furthermore, $|L| \in \{0, 1 \ldots, 2012\}$ and any two tournaments with $|L_1| = |L_2|$ are indistinguishable. Thus, there are 2013 distinct imbalanced tournaments with $|G| = 3$.

If $|G| = 4$, the only distinct $G = \{a_i, a_j, a_k, a_l\}$ is WLOG $a_i \to a_j, a_k; a_j \to a_k, a_l; a_k \to a_l; a_l \to a_j$ and $|L| \in \{0, 1, \ldots, 2011\}$. Similarly, there are 2012 distinct imbalanced tournaments with $|G| = 4$.

If $|G| = 5$, there are 9 distinct graphs G. If each of the teams wins exactly 2 games (within G), then there are 2 distinct G. If three win exactly 2 games, one wins 3, and one wins 1, then there are 3. Finally, if one wins exactly 2 games, two win 3, and two win 1, then there are 4. The distinct graphs G are shown below, but we will omit the procedure for determining them for the sake of clarity. Since $|L| \in \{0, 1, \ldots, 2010\}$, there are $2011 \cdot (2 + 3 + 4) = 18099$ distinct imbalanced tournaments with $|G| = 5$.

In total, then, there are $1 + 2013 + 2012 + 18099 = \boxed{22125}$ distinct imbalanced tournaments.

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