PUMaC 2015

Individual Finals A

1. Alice is placing bishops on a 2015-by-2015 chessboard such that no two can attack one another. (Bishops attack each other if they are on a diagonal.) Her friend Bob notices that he is not able to place down a larger number of bishops such that any two still cannot attack one another. If there are $\prod p_i^{\alpha_i}$, with $\alpha_i > 0$ and $p_i > 0$ prime for all i, ways Alice could have placed her bishops, find $\sum p_i + \alpha_i$.

2. For an odd prime number p, let S denote the following sum taken modulo p:

$$S \equiv \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{(p-2) \cdot (p-1)} \equiv \sum_{i=1}^{\frac{p-1}{2}} \frac{1}{(2i-1) \cdot 2i} \pmod{p}$$

   Prove that $p^2 | 2^p - 2$ if and only if $S \equiv 0 \pmod{p}$.

3. Let I be the incenter of a triangle ABC with $AB = 20$, $BC = 15$, and $BI = 12$. Let CI intersect the circumcircle ω_1 of ABC at $D \neq A$. Alice draws a line l through D that intersects ω_1 on the minor arc AC at X and the circumcircle ω_2 of AIC at Y outside ω_1. She notices that she can construct a right triangle with side lengths ID, DX, and XY. What is the length of IY?