Team Test

Your team receives up to 100 points total for the team round which has 16 questions. However, you will also play a minigame based on your performance that will award you bonus points. Here is the game:

- Let n be number of questions you answer correctly. You will have $\lceil n/4 \rceil$ guesses to make on this game. The game is to pick an integer from $\{1, 2, 3, \ldots, 20\}$.
- Let $x \in \{1, 2, \ldots, 20\}$ be the value you pick. Then the score s_x for picking x is

$$x \cdot \left(\frac{1}{2} - \frac{\text{number of guesses for } x}{\text{total number of guesses}} \right),$$

  where the number of guesses for x is the total number of guesses for x across all teams playing, and the total number of guesses is summed over all teams. Thus your score is linked to how unique of a number you choose, and how high it is.
- Over all of the guesses you make, we choose the highest score s_x, and $\lfloor s_x \rfloor$ is the bonus points you are awarded (if $\lfloor s_x \rfloor$ is negative, your bonus points is instead 0).

Example. *For example, suppose there are only two teams competing: teams A and B.*

- *Suppose team A answers* 7 *questions correct and B answers* 9*. Thus A has* 2 *guesses and B has* 3 *guesses.*
- *Suppose team A guesses* 19 *and* 20*, and team B guesses 18, 19, and 20.*
- *Then* $s_{18} = 18 \cdot (1/2 - 1/5) = 5.4$*,* $s_{19} = 19 \cdot (1/2 - 2/5) = 1.9$*, and* $s_{20} = 20 \cdot (1/2 - 2/5) = 2$.
- *Therefore team A is awarded 2 bonus points and team B is awarded 5 bonus points.*

To play this game, please give us your top 4 choices on the answer sheet, and we will take as many guesses as your team earns after calculating the number of correct answers. E.G if you give us 20, 19, 18, and 17, and you score 6 questions, we will take your first 2 guesses, 20 and 19, to use in this game. Guess 1 is highest priority, down to Guess 4 which is lowest priority.

1. [**2**] Let $f(n)$ denote the sum of the distinct positive integer divisors of n. Evaluate:
$$f(1)+f(2)+f(3)+f(4)+f(5)+f(6)+f(7)+f(8)+f(9).$$

2. [**2**] Sally is going shopping for stuffed tigers. She finds 5 orange, 10 white, and 2 cinnamon colored tigers. Sally decides to buy two tigers of different colors. Assuming all the tigers are distinct, in how many ways can she choose two tigers?

3. [**3**] How many ordered pairs (a, b) of positive integers with $1 \le a, b \le 10$ are there such that in the geometric sequence whose first term is a and whose second term is b, the third term is an integer?

4. [**3**] Ryan is messing with Brice's coin. He weights the coin such that it comes up on one side twice as frequently as the other, and he chooses whether to weight heads or tails more with equal probability. Brice flips his modified coin twice and it lands up heads both times. The probability that the coin lands up heads on the next flip can be expressed in the form $\frac{p}{q}$ for positive integers p, q satisfying $\gcd(p, q) = 1$, what is $p + q$?

5. [**4**] Imagine a regular a 2015-gon with edge length 2. At each vertex, draw a unit circle centered at that vertex and color the circle's circumference orange. Now, another unit circle S is placed inside the polygon such that it is externally tangent to two adjacent circles centered at the vertices. This circle S is allowed to roll freely in the interior of the polygon as long as it remains externally tangent to the vertex circles. As it rolls, S turns the color of any point it touches into black. After it rolls completely around the interior of the polygon, the total length of the black lengths can be expressed in the form $\frac{p\pi}{q}$ for positive integers p, q satisfying $\gcd(p, q) = 1$. What is $p + q$?

6. [**4**] What is the smallest positive integer n such that $2^n - 1$ is a multiple of 2015?

7. [**5**] Charlie noticed his golden ticket was golden in two ways! In addition to being gold, it was a rectangle whose side lengths had ratio the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. He then folds the ticket so that two opposite corners (vertices connected by a diagonal) coincide and makes a sharp crease (the ticket folds just as any regular piece of paper would). The area of the resulting shape can be expressed as $a + b\phi$. What is $\frac{b}{a}$?

8. [**5**] Let $\sigma_1 : \mathbb{N} \to \mathbb{N}$ be a function that takes a natural number n, and returns the sum of the positive integer divisors of n. For example, $\sigma_1(6) = 1 + 2 + 3 + 6 = 12$. What is the largest number n such that $\sigma_1(n) = 1854$?

9. [**6**] Triangle ABC has $\overline{AB} = 5, \overline{BC} = 4, \overline{CA} = 6$. Points D and E are on sides AB and AC, respectively, such that $\overline{AD} = \overline{AE} = \overline{BC}$. Let CD and BE intersect at F and let AF and DE intersect at G. The length of $\overline{FG}$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ in simplified form. What is $a + b + c$?

10. [**7**] Let S be the set of integer triplets (a, b, c) with $1 \le a \le b \le c$ that satisfy $a + b + c = 77$ and:
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{5}.$$
What is the value of the sum $\sum_{(a,b,c)\in S} a \cdot b \cdot c$?

11. [**8**] Given a rational number r that, when expressed in base-10, is a repeating, non-terminating decimal, we define $f(r)$ to be the number of digits in the decimal representation of r that are after the decimal point but before the repeating part of r. For example, $f(1.\overline{27}) = 0$ and $f(0.35\overline{2}) = 2$. What is the smallest positive integer n such that $\frac{1}{n}$, $\frac{2}{n}$, and $\frac{4}{n}$ are non-terminating decimals, where $f\left(\frac{1}{n}\right) = 3$, $f\left(\frac{2}{n}\right) = 2$, and $f\left(\frac{4}{n}\right) = 2$?

12. [**8**] Alice is stacking balls on the ground in three layers using two sizes of balls: small and large. All small balls are the same size, as are all large balls. For the first layer, she uses 6 identical large balls A, B, C, D, E, and F all touching the ground and so that D, E, F touch each other, A touches E and F, B touches D and F, and C touches D and E. For the second layer, she uses 3 identical small balls, G, H, and I; G touches A, E, and F, H touches B, D, and F, and I touches C, D, and E. Obviously, the small balls do not intersect the ground. Finally, for the top layer, she uses one large ball that touches D, E, F, G, H, and I. If the large balls have volume 2015, the sum of the volumes of all the balls in the pyramid can be written in the form $a\sqrt{b}+c$ for integers a, b, c where no integer square larger than 1 divides b. What is $a+b+c$? (This diagram may not have the correct scaling, but just serves to clarify the layout of the problem.)

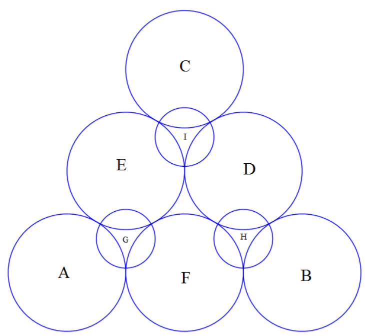

Figure 1: The projection of the balls onto the ground.

13. [**8**] We define $\lfloor x \rfloor$ as the largest integer less than or equal to x. What is

$$\left\lfloor \frac{5^{2017015}}{5^{2015}+7} \right\rfloor \mod 1000?$$

14. [**10**] Marie is painting a 4×4 grid of identical square windows. Initially, they are all orange but she wants to paint 4 of them black. How many ways can she do this up to rotation and reflection?

15. [**12**] Let S be the set of ordered integer pairs (x, y) such that $0 < x < y < 42$ and there exists some integer n such that $x^6 - y^6 \mid n^2 + 2015^2$. What is the sum $\sum_{(x_i,y_i)\in S} x_i y_i$?

16. [**13**] Let p, u, m, a, c be real numbers satisfying $5p^5 + 4u^5 + 3m^5 + 2a^5 + c^5 = 91$. What is the maximum possible value of:

$$18pumac + 2(2+p)^2 + 23(1+ua)^2 + 15(3+mc)^2?$$