PUMaC 2015

Algebra B

1. [**3**] Roy is starting a baking company and decides that he will sell cupcakes. He sells n cupcakes for $(n+20)(n+15)$ cents. A man walks in and buys \$10.50 worth of cupcakes. Roy bakes cupcakes at a rate of 10 cupcakes an hour. How many minutes will it take Roy to complete the order?

2. [**3**] Let f be a function which takes in $0, 1, 2$ and returns $0, 1$, or 2. The values need not be distinct: for instance we could have $f(0) = 1, f(1) = 1, f(2) = 2$. How many such functions are there which satisfy:
$$f(2) + f(f(0)) + f(f(f(1))) = 5$$

3. [**4**] Andrew and Blair are bored in class and decide to play a game. They pick a pair (a, b) with $1 \leq a, b \leq 100$. Andrew says the next number in the geometric series that begins with a, b and Blair says the next number in the arithmetic series that begins with a, b. For how many pairs (a, b) is Andrew's number minus Blair's number a positive perfect square?

4. [**4**] There are real numbers a, b, c, d such that for all (x, y) satisfying $6y^2 = 2x^3 + 3x^2 + x$, if $x_1 = ax + b$ and $y_1 = cy + d$, then $y_1^2 = x_1^3 - 36x_1$. What is $a + b + c + d$?

5. [**5**] Find the sum of the non-repeated roots of the polynomial $P(x) = x^6 - 5x^5 - 4x^4 - 5x^3 + 8x^2 + 7x + 7$.

6. [**6**] Define the sequence a_i as follows: $a_1 = 1, a_2 = 2015$, and $a_n = \frac{na_{n-1}^2}{a_{n-1} + na_{n-2}}$ for $n > 2$. What is the least k such that $a_k < a_{k-1}$?

7. [**7**] We define the function $f(x, y) = x^3 + (y-4)x^2 + (y^2 - 4y + 4)x + (y^3 - 4y^2 + 4y)$. Then choose any distinct $a, b, c \in \mathbb{R}$ such that the following holds: $f(a, b) = f(b, c) = f(c, a)$. Over all such choices of a, b, c, what is the maximum value achieved by:
$$\min(a^4 - 4a^3 + 4a^2, b^4 - 4b^3 + 4b^2, c^4 - 4c^3 + 4c^2)?$$

8. [**8**] We define the ridiculous numbers recursively as follows:

   (a) 1 is a ridiculous number.

   (b) If a is a ridiculous number, then $\sqrt{a}$ and $1 + \sqrt{a}$ are also ridiculous numbers.

   A closed interval I is "boring" if

   - I contains no ridiculous numbers, and
   - There exists an interval $[b, c]$ containing I for which b and c are both ridiculous numbers.

   The smallest non-negative l such that there does not exist a boring interval with length l can be represented in the form $\dfrac{a + b\sqrt{c}}{d}$ where a, b, c, d are integers, $\gcd(a, b, d) = 1$, and no integer square greater than 1 divides c. What is $a + b + c + d$?